Quantum Field Theory: Example Sheet 3

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1. The Weyl representation of the Clifford algebra is given by,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{1}$$

Show that these indeed satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, where $\mathbf{1}$ comes with an implicit 4×4 unit matrix. Find a unitary matrix U such that $(\gamma')^\mu = U\gamma^\mu U^\dagger$, where $(\gamma')^\mu$ form the Dirac representation of the Clifford algebra

$$(\gamma')^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad (\gamma')^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2}$$

2. Show that if $\{\gamma^\mu\, , \, \gamma^\nu\} = 2\eta^{\mu\nu}$, then

$$\left[\gamma^\kappa\gamma^\lambda\, , \, \gamma^\mu\gamma^\nu\right] = 2\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2\eta^{\kappa\mu}\gamma^\lambda\gamma^\nu + 2\eta^{\lambda\nu}\gamma^\mu\gamma^\kappa - 2\eta^{\kappa\nu}\gamma^\mu\gamma^\lambda\, . \tag{3}$$

Show further that $S^{\mu\nu} \equiv \frac{1}{4}\left[\gamma^\mu\, , \, \gamma^\nu\right] = \frac{1}{2}(\gamma^\mu\gamma^\nu - \eta^{\mu\nu})$. Use this to confirm that the matrices $S^{\mu\nu}$ form a representation of the Lie algebra of the Lorentz group.

3. Using just the algebra $\{\gamma^\mu,\ \gamma^\nu\} = 2\eta^{\mu\nu}$ (i.e. without resorting to a particular representation), and defining $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{p} = p_\mu\gamma^\mu$ and $S^{\mu\nu} = \frac{1}{4}\left[\gamma^\mu,\ \gamma^\nu\right]$, prove the following results: (*Some useful tricks include the cyclicity of the trace, and inserting* $(\gamma^5)^2 = 1$ *into a trace*).

i. $\mathrm{Tr}\gamma^\mu = 0$

ii. $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$

iii. $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 0$

iv. $\left(\gamma^5\right)^2 = 1$

v. $\mathrm{Tr}\gamma^5 = 0$

vi. $\not{p}\,\not{q} = 2p\cdot q - \not{q}\,\not{p} = p\cdot q + 2S^{\mu\nu}p_\mu q_\nu$

vii. $\mathrm{Tr}(\not{p}\,\not{q}) = 4p\cdot q$

viii. $\mathrm{Tr}(\not{p}_1 \ldots \not{p}_n) = 0$ if n is odd

ix. $\mathrm{Tr}(\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) = 4\left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4)\right]$

x. $\mathrm{Tr}(\gamma^5 \not{p}_1 \not{p}_2) = 0$

xi. $\gamma_\mu \not{p} \gamma^\mu = -2 \not{p}$

xii. $\gamma_\mu \not{p}_1 \not{p}_2 \gamma^\mu = 4 p_1 \cdot p_2$

xiii. $\gamma_\mu \not{p}_1 \not{p}_2 \not{p}_3 \gamma^\mu = -2 \not{p}_3 \not{p}_2 \not{p}_1$

xiv. $\mathrm{Tr}(\gamma^5 \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) = 4i\, \epsilon_{\mu\nu\rho\sigma}\, p_1^\mu\, p_2^\nu\, p_3^\rho\, p_\sigma^4$

4. The plane-wave solutions to the Dirac equation are

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \quad \text{and} \quad v^s(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \tag{4}$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and ξ^s, with $s = 1, 2$, is a basis of orthonormal two-component spinors, satisfying $(\xi^r)^\dagger \cdot \xi^s = \delta^{rs}$. Show that

$$\begin{aligned} u^r(\vec{p})^\dagger \cdot u^s(\vec{p}) &= 2p_0 \delta^{rs} \\ \bar{u}^r(\vec{p}) \cdot u^s(\vec{p}) &= 2m\delta^{rs} \end{aligned} \tag{5}$$

and similarly,

$$\begin{aligned} v^r(\vec{p})^\dagger \cdot v^s(\vec{p}) &= 2p_0 \delta^{rs} \\ \bar{v}^r(\vec{p}) \cdot v^s(\vec{p}) &= -2m\delta^{rs} \end{aligned} \tag{6}$$

Show also that the orthonality condition between u and v is

$$\bar{u}^s(\vec{p}) \cdot v^r(\vec{p}) = 0 \tag{7}$$

while taking the inner product using $\dagger$ requires an extra minus sign

$$u^r(\vec{p})^\dagger \cdot v^s(-\vec{p}) = 0 \tag{8}$$

5. Using the same notation as Question 4 show that

$$\sum_{s=1}^{2} u^s(\vec{p}) \bar{u}^s(\vec{p}) = \not{p} + m \tag{9}$$

$$\sum_{s=1}^{2} v^s(\vec{p}) \bar{v}^s(\vec{p}) = \not{p} - m \tag{10}$$

where, rather than being contracted, the two spinors on the left-hand side are placed back to back to form a 4×4 matrix.

6. The Fourier decomposition of the Dirac operator $\psi(\vec{x})$ and the conjugate field $\psi^\dagger(\vec{x})$ is given by,

$$\begin{aligned}\psi(\vec{x}) &= \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b_{\vec{p}}^s u^s(\vec{p}) e^{+i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^{s\,\dagger} v^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right] \\ \psi^\dagger(\vec{x}) &= \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b_{\vec{p}}^{s\,\dagger} u^s(\vec{p})^\dagger e^{-i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^s v^s(\vec{p})^\dagger e^{+i\vec{p}\cdot\vec{x}} \right] \end{aligned} \tag{11}$$

The creation and annihilation operators are taken to satisfy

$$\begin{aligned}\{b_{\vec{p}}^r, b_{\vec{q}}^{s\,\dagger}\} &= (2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p} - \vec{q}) \\ \{c_{\vec{p}}^r, c_{\vec{q}}^{s\,\dagger}\} &= (2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p} - \vec{q}) \end{aligned} \tag{12}$$

with all other anti-commutators vanishing,

$$\{b_{\vec{p}}^r, b_{\vec{q}}^s\} = \{c_{\vec{p}}^r, c_{\vec{q}}^s\} = \{b_{\vec{p}}^r, c_{\vec{q}}^{s\,\dagger}\} = \{b_{\vec{p}}^r, c_{\vec{q}}^s\} = \ldots = 0 \tag{13}$$

Show that these imply that the field and it conjugate momenta satisfy the anti-commutation relations,

$$\begin{aligned}\{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} &= \{\psi_\alpha^\dagger(\vec{x}), \psi_\beta^\dagger(\vec{y})\} = 0 \\ \{\psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y})\} &= \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x} - \vec{y}) \end{aligned} \tag{14}$$

(Note: The calculation is very similar to that for the bosonic field, but at some point you will need to make use of the identities (9) and (10)).

7. Using the results of Question 6, show that the quantum Hamiltonian

$$H = \int d^3x\ \bar{\psi}(-i\gamma^i\partial_i + m)\psi \tag{15}$$

can be written, after normal ordering, as

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \sum_{s=1}^{2} \left[b_{\vec{p}}^{s\,\dagger} b_{\vec{p}}^s + c_{\vec{p}}^{s\,\dagger} c_{\vec{p}}^s \right] \tag{16}$$

(Note: Again, the calculation is very similar to that for the bosonic field. This time you will need to make use of the identities derived in Questions 4 and 5).

8. The purpose of this question is to give you a glimpse into the spin-statistics theorem. This theorem roughly says that if you try to quantize a field with the wrong statistics, bad things will happen. Here we'll see what goes wrong if you try to quantize a spin 1/2 field as a boson. We start with the usual decomposition (11). This time we choose bosonic commutation relations for the annihilation and creation operators,

$$\begin{aligned}[][b^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}] &= (2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p}-\vec{q}) \\ [c^r_{\vec{p}}, c^{s\,\dagger}_{\vec{q}}] &= -(2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p}-\vec{q}) \end{aligned} \tag{17}$$

with all other commutators vanishing. Note the strange minus sign for the c operators. Repeat the calculation of Question 6 to show that these are equivalent to the commutation relations,

$$\begin{aligned}[][\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] = [\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})] = 0 \\ [\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})] = \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{y}) \end{aligned} \tag{18}$$

Now repeat the calculation of Question 7, to show that, after normal ordering, the Hamitonian is given by

$$H = \int \frac{d^3p}{(2\pi)^3}\, E_{\vec{p}} \sum_{s=1}^{2} \left[b^{s\,\dagger}_{\vec{p}} b^s_{\vec{p}} - c^{s\,\dagger}_{\vec{p}} c^s_{\vec{p}} \right] \tag{19}$$

This Hamiltonian is not bounded below: you can lower the energy indefinitely by creating more and more c particles. This is the reason a theory of bosonic spin 1/2 particles is sick.