String Theory: Example Sheet 1

Dr David Tong, January 2010

1. Consider an electron in orbit around a neutron. Assume that electromagnetic dipole effects can be neglected and only the Newtonian gravitational force is relevant. What is the radius of Bohr orbit of the ground state? What object is this comparable to? Now re-evaluate your sense of the importance of quantum gravity.

2a. Consider the following action for a point particle

$$S = \frac{1}{2}\int d\tau\ \left(e^{-1}\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - em^2\right) \tag{1}$$

The dynamical fields are $e(\tau)$ and $X^\mu(\tau)$. Write down the equations of motion. Show that after substituting the equation of motion for e, one recovers the form of the point-particle action that we saw in the lectures.

b. Higher-dimensional objects are called *branes*. More precisely, if the object has p spatial dimensions, it is called a p-brane. (Joke copyright Paul Townsend). The dynamics of a p-brane moving in Minkowski space is given by the Dirac action,

$$S = -T\int d^{p+1}\sigma\sqrt{-\det\gamma} \tag{2}$$

Here σ^α, $\alpha = 0,\ldots,p$ are coordinates on the brane worldvolume, while $\gamma_{\alpha\beta}$ is the pull-back of the Minkowski metric onto the brane.

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu} \tag{3}$$

Show that this is equivalent to the Polyakov-type action with dynamical worldvolume metric $g_{\alpha\beta}$,

$$S = -\frac{T}{2}\int d^{p+1}\sigma\ \sqrt{-g}\left(g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} - (p-1)\right)$$

3. Show that the Polyakov action for the string is reparameterization invariant.

4. The Polyakov string is invariant under translational and Lorentz symmetries,

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu$$

Show that, in conformal gauge, the Noether currents associated to these global symmetries on the worldsheet are

$$P^\alpha_\mu = T\partial^\alpha X_\mu \quad \text{and} \quad J^\alpha_{\mu\nu} = P^\alpha_\mu X_\nu - P^\alpha_\nu X_\mu$$

Write down the conserved charge associated with Lorentz transformations in terms of the modes of the string and interpret the result.

5. By considering the appropriate Noether charge, show that p^μ in the mode expansion has the interpretation of the center of mass momentum of the string. Do this for both open and closed strings.

6. The purpose of this exercise is to get some feel for the properties of classical solutions to the string equation of motion. Throughout this exercise, it will be useful to work in static gauge $X^0 = R\tau$.

a. Suppose that at a fixed time, a snapshot of a closed string looks like a circle lying in the $X^1 - X^2$ plane. Write down an instantaneous configuration describing the string that obeys the constraint equations. By using the Noether current, compute the energy of this configuration.

b. Write down a configuration describing a spinning, circular closed string lying in the $X^1 - X^2$ plane which obeys the equation of motion. Show that this does *not* obey the constraints. What is this telling you?

c. Show that the ends of an open string with only Neumann boundary conditions travel at the speed of light.

d. A closed string may form a cusp. If we require that the worldsheet fields remain smooth, we must have $\vec{x}'(\sigma,\tau) = 0$ at the cusp. Show that this point of the string travels with luminal velocity, perpendicular to the direction of the cusp. Show that such cusps are generic for a classical string moving in $\mathbf{R}^{1,3}$, but not in higher dimensions.

7. Show that the Ricci curvature for the the conformally flat 2d Euclidean metric $g_{\alpha\beta} = e^{2\phi}\delta_{\alpha\beta}$ is given by

$$R = -2e^{-2\phi}\,\partial^2\phi$$

8. Assuming the commutation relations for x^μ, p^μ, α_n^μ and $\tilde{\alpha}_n^\mu$ in the mode expansion, compute the commutation relations for X^μ and Π^μ.

9. Construct the open string states at level 2 in the lightcone formalism and determine their representation under $SO(D-1)$. Construct the states at level 3. Show that they fit into a traceless symmetric-3-tensor and an anti-symmetric-2-tensor representation of $SO(D-1)$.

10. Write down the mode expansion describing an open string stretched between two parallel Dp-branes. Interpret the result.

11. The purpose of this exercise is to derive some formulae that are important in the covariant quantization of the string. Later we see these same formulae appearing in the context of conformal field theory. You might find the book by Green, Schwarz and Witten useful.

Using the commutation relations for α_n^μ, show that the normal ordered Virasoro generators satisfy

$$[L_m, L_n] = \frac{1}{2}\sum_p p\,\alpha_{m-p}\cdot\alpha_{p+n} + (m-p)\alpha_{n+m-p}\cdot\alpha_p$$

By swapping dummy variables, make an argument that the L_n obey

$$[L_m, L_n] = (m-n)L_{m+n}$$

Why do you think the argument leading to this result might be suspect in the quantum theory? Convince yourself that the correct form of the commutation relations is

$$[L_m, L_n] = (m-n)L_{m+n} + C(n)\delta_{m+n,0}$$

for some (as yet undetermined) real-valued function $C(n)$ such that $C(-n) = -C(n)$. This extra piece is called the central extension of the algebra.

By considering the Jacobi identity, show that the function $C(n)$ can take the form,

$$C(n) = c_3 n^3 + c_1 n$$

for some c_3 and c_1.

Compute the matrix elements,

$$\langle 0; p | [L_1, L_{-1}] | 0; p \rangle \quad \text{and} \quad \langle 0; p | [L_2, L_{-2}] | 0; p \rangle$$

Use this to show that the correct commutation relations are given by the *Virasoro algebra*,

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0}$$