String Theory: Example Sheet 2

Dr David Tong, February 2010 *"Wax on. Wax off."*

1. Verify that

$$\partial\bar{\partial}\ \ln|z|^2 = 2\pi\delta(z,\bar{z})\ ,$$

firstly by using the divergence theorem, and secondly by regulating the singularity at $z = 0$.

2. Show that $:e^{ikX}:$ is a primary operator for the theory of a free scalar field. Compute the weight.

3. A theory of a free scalar field X has OPE

$$\partial X(z)\,\partial X(w) = -\frac{\alpha'}{2}\frac{1}{(z-w)^2} + \ldots$$

Consider a putative candidate for the stress-energy tensor

$$T(z) = -\frac{1}{\alpha'} : \partial X(z)\partial X(z) : -Q\,\partial^2 X(z)$$

a. Use the TX OPE to determine the transformation of X under conformal transformations $\delta z = \epsilon(z)$.

b. Show that ∂X is not primary unless $Q = 0$, but is quasi-primary with weight $h = 1$. Show that $:e^{ikX}:$ is primary and compute its weight.

c. Determine the TT OPE and show that it does have the structure expected of a stress-energy tensor. What is the central charge of the theory?

An Aside: What's going on here? How can the same free scalar field have different stress-energy tensors? The point is that the stress energy tensor tells us how the system couples to a curved metric. Several such couplings could all give the same physics when restricted to a flat background. The theory considered here is called the *linear dilaton* theory. We will see where it comes from later in the course.

4. A theory of several free, non-interacting scalars X^μ, $\mu = 1, \ldots D$, has the operators

$$\zeta_\mu : \partial X^\mu\, e^{ik\cdot X} : \qquad \text{and} \qquad \zeta_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu\, e^{ik\cdot X} :$$

where ζ_μ, k_μ are constant vectors and $\zeta_{\mu\nu}$ is a constant tensor. If the stress-energy tensor is given by $T = (-1/\alpha') \, :\partial X^\mu \partial X_\mu:$ determine the conditions for these operators to be primary. What are their weights?

5. A free fermion Majorana fermion in two dimensions has action,

$$S = \frac{1}{2\pi} \int d^2z \; \psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}$$

The propagator is given by the OPE

$$\psi(z)\psi(w) = -\psi(w)\psi(z) = \frac{1}{z-w}$$

and similar for $\bar{\psi}$. (Remember, ψ and $\bar{\psi}$ are Grassmann-valued fields, a fact which is reflected in the OPE). The energy momentum tensor is

$$T_{zz} = -\frac{1}{2} : \psi \partial \psi :$$

Show that ψ is a primary operator of weight $1/2$. Determine the central charge of this theory.

6. The bc ghost system consist of two free Grassmann fields b and c. (Note: do not confuse the field c with the central charge c. They are not the same thing!) The OPE is given by

$$b(z)c(w) = -c(w)b(z) = \frac{1}{z-w}$$

Consider the stress-energy tensor

$$T =: (\partial b)c : -\lambda \partial : bc :$$

Show that b is primary with weight $h = \lambda$ and c is primary with weight $h = 1 - \lambda$. Show that the central charge of this system is equal to

$$c = -12\lambda^2 + 12\lambda - 2$$

An Aside: This peculiar looking theory is *extremely* important. We will come across it later in the course when we discuss the path integral approach to string theory.

7. Show that the Schwarzian transformation of the stress tensor reproduces the correct infinitesimal transformation. Show, moreover, that it has the correct property under successive conformal transformations.

8. Use the OPE for a free scalar field to determine the commutation relations of the Fourier modes α_m.