String Theory

University of Cambridge Part III Mathematical Tripos

Dr David Tong

Department of Applied Mathematics and Theoretical Physics,
Centre for Mathematical Sciences,
Wilberforce Road,
Cambridge, CB3 OWA, UK

http://www.damtp.cam.ac.uk/user/tong/string.html
d.tong@damtp.cam.ac.uk

Recommended Books and Resources

- J. Polchinski, *String Theory*

This two volume work is the standard introduction to the subject. Our lectures will more or less follow the path laid down in volume one covering the bosonic string. The book contains explanations and descriptions of many details that have been deliberately (and, I suspect, at times inadvertently) swept under a very large rug in these lectures. Volume two covers the superstring.

- M. Green, J. Schwarz and E. Witten, *Superstring Theory*

Another two volume set. It is now over 20 years old and takes a slightly old-fashioned route through the subject, with no explicit mention of conformal field theory. However, it does contain much good material and the explanations are uniformly excellent. Volume one is most relevant for these lectures.

- B. Zwiebach, *A First Course in String Theory*

This book grew out of a course given to undergraduates who had no previous exposure to general relativity or quantum field theory. It has wonderful pedagogical discussions of the basics of lightcone quantization. More surprisingly, it also has some very clear descriptions of several advanced topics, even though it misses out all the bits in between.

- P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*

This big yellow book is affectionately known as the yellow pages. It's a great way to learn conformal field theory. At first glance, it comes across as slightly daunting because it's big. (And yellow). But you soon realise that it's big because it starts at the beginning and provides detailed explanations at every step. The material necessary for this course can be found in chapters 5 and 6.

Further References: "*String Theory and M-Theory*" by Becker, Becker and Schwarz and "*String Theory in a Nutshell*" (it's a big nutshell) by Kiritsis both deal with the bosonic string fairly quickly, but include more advanced topics that may be of interest. The book "*D-Branes*" by Johnson has lively and clear discussions about the many joys of D-branes. Links to several excellent online resources, including video lectures by Shiraz Minwalla, are listed on the course webpage.

Contents

Acknowledgements

0. Introduction

String theory is an ambitious project. It purports to be an all-encompassing theory of the universe, unifying the forces of nature, including gravity, in a single quantum mechanical framework.

The premise of string theory is that, at the fundamental level, matter does not consist of point-particles but rather of tiny loops of string. From this slightly absurd beginning, the laws of physics emerge. General relativity, electromagnetism and Yang-Mills gauge theories all appear in a surprising fashion. However, they come with baggage. String theory gives rise to a host of other ingredients, most strikingly extra spatial dimensions of the universe beyond the three that we have observed. The purpose of this course is to understand these statements in detail.

These lectures differ from most other courses that you will take in a physics degree. String theory is speculative science. There is no experimental evidence that string theory is the correct description of our world and scant hope that hard evidence will arise in the near future. Moreover, string theory is very much a work in progress and certain aspects of the theory are far from understood. Unresolved issues abound and it seems likely that the final formulation has yet to be written. For these reasons, I'll begin this introduction by suggesting some answers to the question: Why study string theory?

Reason 1. String theory is a theory of quantum gravity

String theory unifies Einstein's theory of general relativity with quantum mechanics. Moreover, it does so in a manner that retains the explicit connection with both quantum theory and the low-energy description of spacetime.

But quantum gravity contains many puzzles, both technical and conceptual. What does spacetime look like at the shortest distance scales? How can we understand physics if the causal structure fluctuates quantum mechanically? Is the big bang truely the beginning of time? Do singularities that arise in black holes really signify the end of time? What is the microscopic origin of black hole entropy and what is it telling us? What is the resolution to the information paradox? Some of these issues will be reviewed later in this introduction.

Whether or not string theory is the true description of reality, it offers a framework in which one can begin to explore these issues. For some questions, string theory has given very impressive and compelling answers. For others, string theory has been almost silent.

Reason 2. String theory may be *the* theory of quantum gravity

With broad brush, string theory looks like an extremely good candidate to describe the real world. At low-energies it naturally gives rise to general relativity, gauge theories, scalar fields and chiral fermions. In other words, it contains all the ingredients that make up our universe. It also gives the only presently credible explanation for the value of the cosmological constant although, in fairness, I should add that the explanation is so distasteful to some that the community is rather amusingly split between whether this is a good thing or a bad thing. Moreover, string theory incorporates several ideas which do not yet have experimental evidence but which are considered to be likely candidates for physics beyond the standard model. Prime examples are supersymmetry and axions.

However, while the broad brush picture looks good, the finer details have yet to be painted. String theory does not provide unique predictions for low-energy physics but instead offers a bewildering array of possibilities, mostly dependent on what is hidden in those extra dimensions. Partly, this problem is inherent to any theory of quantum gravity: as we'll review shortly, it's a long way down from the Planck scale to the domestic energy scales explored at the LHC. Using quantum gravity to extract predictions for particle physics is akin to using QCD to extract predictions for how coffee makers work. But the mere fact that it's hard is little comfort if we're looking for convincing evidence that string theory describes the world in which we live.

While string theory cannot at present offer falsifiable predictions, it has nonetheless inspired new and imaginative proposals for solving outstanding problems in particle physics and cosmology. There are scenarios in which string theory might reveal itself in forthcoming experiments. Perhaps we'll find extra dimensions at the LHC, perhaps we'll see a network of fundamental strings stretched across the sky, or perhaps we'll detect some feature of non-Gaussianity in the CMB that is characteristic of D-branes at work during inflation. My personal feeling however is that each of these is a long shot and we may not know whether string theory is right or wrong within our lifetimes. Of course, the history of physics is littered with naysayers, wrongly suggesting that various theories will never be testable. With luck, I'll be one of them.

Reason 3. String theory provides new perspectives on gauge theories

String theory was born from attempts to understand the strong force. Almost forty years later, this remains one of the prime motivations for the subject. String theory provides tools with which to analyze down-to-earth aspects of quantum field theory that are far removed from high-falutin' ideas about gravity and black holes.

Of immediate relevance to this course are the pedagogical reasons to invest time in string theory. At heart, it is the study of conformal field theory and gauge symmetry. The techniques that we'll learn are not isolated to string theory, but apply to countless systems which have direct application to real world physics.

On a deeper level, string theory provides new and very surprising methods to understand aspects of quantum gauge theories. Of these, the most startling is the *AdS/CFT correspondence*, first conjectured by Juan Maldacena, which gives a relationship between strongly coupled quantum field theories and gravity in higher dimensions. These ideas have been applied in areas ranging from nuclear physics to condensed matter physics and have provided qualitative (and arguably quantitative) insights into strongly coupled phenomena.

Reason 4. String theory provides new results in mathematics

For the past 250 years, the close relationship between mathematics and physics has been almost a one-way street: physicists borrowed many things from mathematicians but, with a few noticeable exceptions, gave little back. In recent times, that has changed. Ideas and techniques from string theory and quantum field theory have been employed to give new "proofs" and, perhaps more importantly, suggest new directions and insights in mathematics. The most well known of these is *mirror symmetry*, a relationship between topologically different Calabi-Yau manifolds.

The four reasons described above also crudely characterize the string theory community: there are "relativists" and "phenomenologists" and "field theorists" and "mathematicians". Of course, the lines between these different sub-disciplines are not fixed and one of the great attractions of string theory is its ability to bring together people working in different areas — from cosmology to condensed matter to pure mathematics — and provide a framework in which they can profitably communicate. In my opinion, it is this cross-fertilization between fields which is the greatest strength of string theory.

0.1 Quantum Gravity

This is a starter course in string theory. Our focus will be on the perturbative approach to the bosonic string and, in particular, why this gives a consistent theory of quantum gravity. Before we leap into this, it is probably best to say a few words about quantum gravity itself. Like why it's hard. And why it's important. (And why it's not).

The Einstein Hilbert action is given by

$$S_{EH} = \frac{1}{16\pi G_N} \int d^4 x \sqrt{-g} \mathcal{R}$$

Newton's constant G_N can be written as

$$8\pi G_N = \frac{\hbar c}{M_{pl}^2}$$

Throughout these lectures we work in units with $\hbar = c = 1$. The Planck mass M_{pl} defines an energy scale

$$M_{pl} \approx 2 \times 10^{18} \text{ GeV} .$$

(This is sometimes referred to as the reduced Planck mass, to distinguish it from the scale without the factor of 8π, namely $\sqrt{1/G_N} \approx 1 \times 10^{19}$ GeV).

There are a couple of simple lessons that we can already take from this. The first is that the relevant coupling in the quantum theory is $1/M_{pl}$. To see that this is indeed the case from the perspective of the action, we consider small perturbations around flat Minkowski space,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{pl}} h_{\mu\nu}$$

The factor of $1/M_{pl}$ is there to ensure that when we expand out the Einstein-Hilbert action, the kinetic term for h is canonically normalized, meaning that it comes with no powers of M_{pl}. This then gives the kind of theory that you met in your first course on quantum field theory, albeit with an infinite series of interaction terms,

$$S_{EH} = \int d^4x \, (\partial h)^2 + \frac{1}{M_{pl}} h \, (\partial h)^2 + \frac{1}{M_{pl}^2} h^2 \, (\partial h)^2 + \ldots$$

Each of these terms is schematic: if you were to do this explicitly, you would find a mess of indices contracted in different ways. We see that the interactions are suppressed by powers of M_{pl}. This means that quantum perturbation theory is an expansion in the dimensionless ratio E^2/M_{pl}^2, where E is the energy associated to the process of interest. We learn that gravity is weak, and therefore under control, at low-energies. But gravitational interactions become strong as the energy involved approaches the Planck scale. In the language of the renormalization group, couplings of this type are known as *irrelevant*.

The second lesson to take away is that the Planck scale M_{pl} is very very large. The LHC will probe the electroweak scale, $M_{EW} \sim 10^3$ GeV. The ratio is $M_{EW}/M_{pl} \sim 10^{-15}$. For this reason, quantum gravity will not affect your daily life, even if your daily life involves the study of the most extreme observable conditions in the universe.

Gravity is Non-Renormalizable

Quantum field theories with irrelevant couplings are typically ill-behaved at high-energies, rendering the theory ill-defined. Gravity is no exception. Theories of this type are called *non-renormalizable*, which means that the divergences that appear in the Feynman diagram expansion cannot be absorbed by a finite number of counterterms. In pure Einstein gravity, the symmetries of the theory are enough to ensure that the one-loop S-matrix is finite. The first divergence occurs at two-loops and requires the introduction of a counterterm of the form,

$$\Gamma \sim \frac{1}{\epsilon} \frac{1}{M_{pl}^4} \int d^4x \sqrt{-g} \, \mathcal{R}^{\mu\nu}{}_{\rho\sigma} \mathcal{R}^{\rho\sigma}{}_{\lambda\kappa} \mathcal{R}^{\lambda\kappa}{}_{\mu\nu}$$

with $\epsilon = 4 - D$. All indications point towards the fact that this is the first in an infinite number of necessary counterterms.

Coupling gravity to matter requires an interaction term of the form,

$$S_{int} = \int d^4x \, \frac{1}{M_{pl}} \, h_{\mu\nu} T^{\mu\nu} + \mathcal{O}(h^2)$$

This makes the situation marginally worse, with the first divergence now appearing at one-loop. The Feynman diagram in the figure shows particle scattering through the exchange of two gravitons. When the momentum k running in the loop is large, the diagram is badly divergent: it scales as

$$\frac{1}{M_{pl}^4} \int^\infty d^4k$$
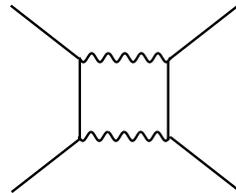
Figure 1:

Non-renormalizable theories are commonplace in the history of physics, the most commonly cited example being Fermi's theory of the weak interaction. The first thing to say about them is that they are far from useless! Non-renormalizable theories are typically viewed as *effective* field theories, valid only up to some energy scale Λ. One deals with the divergences by simply admitting ignorance beyond this scale and treating Λ as a UV cut-off on any momentum integral. In this way, we get results which are valid to an accuracy of E/Λ (perhaps raised to some power). In the case of the weak interaction, Fermi's theory accurately predicts physics up to an energy scale of $\sqrt{1/G_F} \sim 100$ GeV. In the case of quantum gravity, Einstein's theory works to an accuracy of $(E/M_{pl})^2$.

However, non-renormalizable theories are typically unable to describe physics at their cut-off scale Λ or beyond. This is because they are missing the true ultra-violet degrees of freedom which tame the high-energy behaviour. In the case of the weak force, these new degrees of freedom are the W and Z bosons. We would like to know what missing degrees of freedom are needed to complete gravity.

Singularities

Only a particle physicist would phrase all questions about the universe in terms of scattering amplitudes. In general relativity we typically think about the geometry as a whole, rather than bastardizing the Einstein-Hilbert action and discussing perturbations around flat space. In this language, the question of high-energy physics turns into one of short distance physics. Classical general relativity is not to be trusted in regions where the curvature of spacetime approaches the Planck scale and ultimately becomes singular. A quantum theory of gravity should resolve these singularities.

The question of spacetime singularities is morally equivalent to that of high-energy scattering. Both probe the ultra-violet nature of gravity. A spacetime geometry is made of a coherent collection of gravitons, just as the electric and magnetic fields in a laser are made from a collection of photons. The short distance structure of spacetime is governed – after Fourier transform – by high momentum gravitons. Understanding spacetime singularities and high-energy scattering are different sides of the same coin.

There are two situations in general relativity where singularity theorems tell us that the curvature of spacetime gets large: at the big bang and in the center of a black hole. These provide two of the biggest challenges to any putative theory of quantum gravity.

Gravity is Subtle

It is often said that general relativity contains the seeds of its own destruction. The theory is unable to predict physics at the Planck scale and freely admits to it. Problems such as non-renormalizability and singularities are, in a Rumsfeldian sense, known unknowns. However, the full story is more complicated and subtle. On the one hand, the issue of non-renormalizability may not quite be the crisis that it first appears. On the other hand, some aspects of quantum gravity suggest that general relativity isn't as honest about its own failings as is usually advertised. The theory hosts a number of unknown unknowns, things that we didn't even know that we didn't know. We won't have a whole lot to say about these issues in this course, but you should be aware of them. Here I mention only a few salient points.

Firstly, there is a key difference between Fermi's theory of the weak interaction and gravity. Fermi's theory was unable to provide predictions for any scattering process at energies above $\sqrt{1/G_F}$. In contrast, if we scatter two objects at extremely high-energies in gravity — say, at energies $E \gg M_{pl}$ — then we know exactly what will happen: we form a big black hole. We don't need quantum gravity to tell us this. Classical general relativity is sufficient. If we restrict attention to scattering, the crisis of non-renormalizability is not problematic at ultra-high energies. It's troublesome only within a window of energies around the Planck scale.

Similar caveats hold for singularities. If you are foolish enough to jump into a black hole, then you're on your own: without a theory of quantum gravity, no one can tell you what fate lies in store at the singularity. Yet, if you are smart and stay outside of the black hole, you'll be hard pushed to see any effects of quantum gravity. This is because Nature has conspired to hide Planck scale curvatures from our inquisitive eyes. In the case of black holes this is achieved through cosmic censorship which is a conjecture in classical general relativity that says singularities are hidden behind horizons. In the case of the big bang, it is achieved through inflation, washing away any traces from the very early universe. Nature appears to shield us from the effects of quantum gravity, whether in high-energy scattering or in singularities. I think it's fair to say that no one knows if this conspiracy is pointing at something deep, or is merely inconvenient for scientists trying to probe the Planck scale.

While horizons may protect us from the worst excesses of singularities, they come with problems of their own. These are the unknown unknowns: difficulties that arise when curvatures are small and general relativity says "trust me". The entropy of black holes and the associated paradox of information loss strongly suggest that local quantum field theory breaks down at macroscopic distance scales. Attempts to formulate quantum gravity in de Sitter space, or in the presence of eternal inflation, hint at similar difficulties. Ideas of holography, black hole complimentarity and the AdS/CFT correspondence all point towards non-local effects and the emergence of spacetime. These are the deep puzzles of quantum gravity and their relationship to the ultra-violet properties of gravity is unclear.

As a final thought, let me mention the one observation that has an outside chance of being related to quantum gravity: the cosmological constant. With an energy scale of $\Lambda \sim 10^{-3}$ eV it appears to have little to do with ultra-violet physics. If it does have its origins in a theory of quantum gravity, it must either be due to some subtle "unknown unknown", or because it is explained away as an environmental quantity as in string theory.

Is the Time Ripe?

Our current understanding of physics, embodied in the standard model, is valid up to energy scales of 10^3 GeV. This is 15 orders of magnitude away from the Planck scale. Why do we think the time is now ripe to tackle quantum gravity? Surely we are like the ancient Greeks arguing about atomism. Why on earth do we believe that we've developed the right tools to even address the question?

The honest answer, I think, is hubris.

However, there is mild circumstantial evidence that the framework of quantum field theory might hold all the way to the Planck scale without anything very dramatic happening in between. The main argument is unification. The three coupling constants of Nature run logarithmically, meeting miraculously at the GUT energy scale of 10^{15} GeV. Just slightly later, the fourth force of Nature, gravity, joins them. While not overwhelming, this does provide a hint that perhaps quantum field theory can be taken seriously at these ridiculous scales.

Figure 2:

Historically I suspect this was what convinced large parts of the community that it was ok to speak about processes at 10^{18} GeV.

Finally, perhaps the most compelling argument for studying physics at the Planck scale is that string theory *does* provide a consistent unified quantum theory of gravity and the other forces. Given that we have this theory sitting in our laps, it would be foolish not to explore its consequences. The purpose of these lecture notes is to begin this journey.
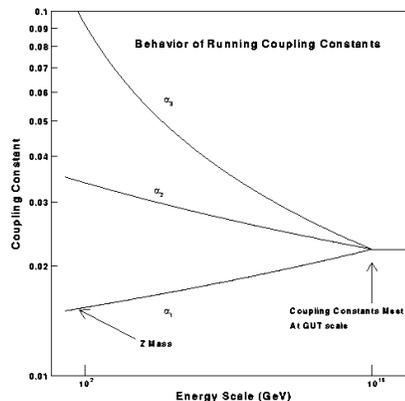
1. The Relativistic String

All lecture courses on string theory start with a discussion of the point particle. Ours is no exception. We'll take a flying tour through the physics of the relativistic point particle and extract a couple of important lessons that we'll take with us as we move onto string theory.

1.1 The Relativistic Point Particle

We want to write down the Lagrangian describing a relativistic particle of mass m. In anticipation of string theory, we'll consider D-dimensional Minkowski space $\mathbf{R}^{1,D-1}$. Throughout these notes, we work with signature

$$\eta_{\mu\nu} = \text{diag}(-1, +1, +1, \ldots, +1)$$

Note that this is the opposite signature to my quantum field theory notes.

If we fix a frame with coordinates $X^\mu = (t, \vec{x})$ the action is simple:

$$S = -m \int dt \sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}} \, . \tag{1.1}$$

To see that this is correct we can compute the momentum $\vec{p}$, conjugate to $\vec{x}$, and the energy E which is equal to the Hamiltonian,

$$\vec{p} = \frac{m\dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}} \quad , \quad E = \sqrt{m^2 + \vec{p}^2} \, ,$$

both of which should be familiar from courses on special relativity.

Although the Lagrangian (1.1) is correct, it's not fully satisfactory. The reason is that time t and space $\vec{x}$ play very different roles in this Lagrangian. The position $\vec{x}$ is a dynamical degree of freedom. In contrast, time t is merely a parameter providing a label for the position. Yet Lorentz transformations are supposed to mix up t and $\vec{x}$ and such symmetries are not completely obvious in (1.1). Can we find a new Lagrangian in which time and space are on equal footing?

One possibility is to treat both time and space as labels. This leads us to the concept of field theory. However, in this course we will be more interested in the other possibility: we will promote time to a dynamical degree of freedom. At first glance, this may appear odd: the number of degrees of freedom is one of the crudest ways we have to characterize a system. We shouldn't be able to add more degrees of freedom

at will without fundamentally changing the system that we're talking about. Another way of saying this is that the particle has the option to move in space, but it doesn't have the option to move in time. It *has* to move in time. So we somehow need a way to promote time to a degree of freedom without it really being a true dynamical degree of freedom! How do we do this? The answer, as we will now show, is gauge symmetry.

Consider the action,

$$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}} \,, \qquad\qquad (1.2)$$

where $\mu = 0, \ldots, D-1$ and $\dot{X}^\mu = dX^\mu/d\tau$. We've introduced a new parameter τ which labels the position along the worldline of the particle as shown by the dashed lines in the figure. This action has a simple interpretation: it is just the proper time $\int ds$ along the worldline.
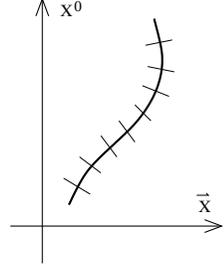
Figure 3:

Naively it looks as if we now have D physical degrees of freedom rather than $D-1$ because, as promised, the time direction $X^0 \equiv t$ is among our dynamical variables: $X^0 = X^0(\tau)$. However, this is an illusion. To see why, we need to note that the action (1.2) has a very important property: reparameterization invariance. This means that we can pick a different parameter $\tilde{\tau}$ on the worldline, related to τ by any monotonic function

$$\tilde{\tau} = \tilde{\tau}(\tau) \,.$$

Let's check that the action is invariant under transformations of this type. The integration measure in the action changes as $d\tau = d\tilde{\tau} \, |d\tau/d\tilde{\tau}|$. Meanwhile, the velocities change as $dX^\mu/d\tau = (dX^\mu/d\tilde{\tau}) \, (d\tilde{\tau}/d\tau)$. Putting this together, we see that the action can just as well be written in the $\tilde{\tau}$ reparameterization,

$$S = -m \int d\tilde{\tau} \sqrt{-\frac{dX^\mu}{d\tilde{\tau}} \frac{dX^\nu}{d\tilde{\tau}} \eta_{\mu\nu}} \,.$$

The upshot of this is that not all D degrees of freedom X^μ are physical. For example, suppose you find a solution to this system, so that you know how X^0 changes with τ and how X^1 changes with τ and so on. Not all of that information is meaningful because τ itself is not meaningful. In particular, we could use our reparameterization invariance to simply set

$$\tau = X^0(\tau) \equiv t \qquad\qquad (1.3)$$

If we plug this choice into the action (1.2) then we recover our initial action (1.1). The reparameterization invariance is a *gauge symmetry* of the system. Like all gauge symmetries, it's not really a symmetry at all. Rather, it is a redundancy in our description. In the present case, it means that although we seem to have D degrees of freedom X^μ, one of them is fake.

The fact that one of the degrees of freedom is a fake also shows up if we look at the momenta,

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{m \dot{X}^\nu \eta_{\mu\nu}}{\sqrt{-\dot{X}^\lambda \dot{X}^\rho \, \eta_{\lambda\rho}}} \tag{1.4}$$

These momenta aren't all independent. They satisfy

$$p_\mu p^\mu + m^2 = 0 \tag{1.5}$$

This is a constraint on the system. It is, of course, the mass-shell constraint for a relativistic particle of mass m. From the worldline perspective, it tells us that the particle isn't allowed to sit still in Minkowski space: at the very least, it had better keep moving in a timelike direction with $(p^0)^2 \geq m^2$.

One advantage of the action (1.2) is that the Poincaré symmetry of the particle is now manifest, appearing as a global symmetry on the worldline

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \tag{1.6}$$

where Λ is a Lorentz transformation satisfying $\Lambda^\mu{}_\nu \eta^{\nu\rho} \Lambda^\sigma{}_\rho = \eta^{\mu\sigma}$, while c^μ corresponds to a constant translation. We have made all the symmetries manifest at the price of introducing a gauge symmetry into our system. A similar gauge symmetry will arise in the relativistic string and much of this course will be devoted to understanding its consequences.

1.1.1 Quantization

It's a trivial matter to quantize this action. We introduce a wavefunction $\Psi(X)$. This satisfies the usual Schrödinger equation,

$$i \frac{\partial \Psi}{\partial \tau} = H \Psi \ .$$

But, computing the Hamiltonian $H = \dot{X}^\mu p_\mu - L$, we find that it vanishes: $H = 0$. This shouldn't be surprising. It is simply telling us that the wavefunction doesn't depend on

τ. Since the wavefunction is something physical while, as we have seen, τ is not, this is to be expected. Note that this doesn't mean that time has dropped out of the problem. On the contrary, in this relativistic context, time X^0 is an operator, just like the spatial coordinates $\vec{x}$. This means that the wavefunction Ψ is immediately a function of space and time. It is not like a static state in quantum mechanics, but more akin to the fully integrated solution to the non-relativistic Schrödinger equation.

The classical system has a constraint given by (1.5). In the quantum theory, we impose this constraint as an operator equation on the wavefunction, namely $(p^\mu p_\mu + m^2)\Psi = 0$. Using the usual representation of the momentum operator $p_\mu = -i\partial/\partial X^\mu$, we recognize this constraint as the Klein-Gordon equation

$$\left(-\frac{\partial}{\partial X^\mu} \frac{\partial}{\partial X^\nu} \eta^{\mu\nu} + m^2 \right) \Psi(X) = 0 \tag{1.7}$$

Although this equation is familiar from field theory, it's important to realize that the interpretation is somewhat different. In relativistic field theory, the Klein-Gordon equation is the equation of motion obeyed by a scalar field. In relativistic quantum mechanics, it is the equation obeyed by the wavefunction. In the early days of field theory, the fact that these two equations are the same led people to think one should view the wavefunction as a classical field and quantize it a second time. This isn't correct, but nonetheless the language has stuck and it is common to talk about the point particle perspective as "first quantization" and the field theory perspective as "second quantization".

So far we've considered only a free point particle. How can we introduce interactions into this framework? We would have to first decide which interactions are allowed: perhaps the particle can split into two; perhaps it can fuse with other particles? Obviously, there is a huge range of options for us to choose from. We would then assign amplitudes for these processes to happen. There would be certain restrictions coming from the requirement of unitarity which, among other things, would lead to the necessity of anti-particles. We could draw diagrams associated to the different interactions — an example is given in the figure — and in this manner we would slowly build up the Feynman diagram expansion that is familiar from field theory. In fact, this was pretty much the way Feynman himself approached the topic of QED. However, in practice we rarely construct particle interactions in this way because the field theory framework provides a much better way of looking at things. In contrast, this way of building up interactions is exactly what we will later do for strings.
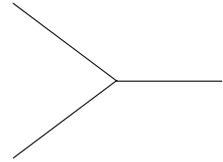
Figure 4:

1.1.2 Ein Einbein

There is another action that describes the relativistic point particle. We introduce yet another field on the worldline, $e(\tau)$, and write

$$S = \frac{1}{2} \int d\tau \, \left(e^{-1} \dot{X}^2 - em^2 \right) , \qquad (1.8)$$

where we've used the notation $\dot{X}^2 = \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}$. For the rest of these lectures, terms like X^2 will always mean an implicit contraction with the spacetime Minkowski metric.

This form of the action makes it look as if we have coupled the worldline theory to 1d gravity, with the field $e(\tau)$ acting as an einbein (in the sense of vierbeins that are introduced in general relativity). To see this, note that we could change notation and write this action in the more suggestive form

$$S = -\frac{1}{2} \int d\tau \, \sqrt{-g_{\tau\tau}} \left(g^{\tau\tau} \dot{X}^2 + m^2 \right) . \qquad (1.9)$$

where $g_{\tau\tau} = (g^{\tau\tau})^{-1}$ is the metric on the worldline and $e = \sqrt{-g_{\tau\tau}}$

Although our action appears to have one more degree of freedom, e, it can be easily checked that it has the same equations of motion as (1.2). The reason for this is that e is completely fixed by its equation of motion, $\dot{X}^2 + e^2 m^2 = 0$. Substituting this into the action (1.8) recovers (1.2)

The action (1.8) has a couple of advantages over (1.2). Firstly, it works for massless particles with $m = 0$. Secondly, the absence of the annoying square root means that it's easier to quantize in a path integral framework.

The action (1.8) retains invariance under reparameterizations which are now written in a form that looks more like general relativity. For transformations parameterized by an infinitesimal η, we have

$$\tau \to \tilde{\tau} = \tau - \eta(\tau) \quad , \quad \delta e = \frac{d}{d\tau}(\eta(\tau)e) \quad , \quad \delta X^\mu = \frac{dX^\mu}{d\tau}\eta(\tau) \qquad (1.10)$$

The einbein e transforms as a density on the worldline, while each of the coordinates X^μ transforms as a worldline scalar.

1.2 The Nambu-Goto Action

A particle sweeps out a worldline in Minkowski space. A string sweeps out a *worldsheet*. We'll parameterize this worldsheet by one timelike coordinate τ, and one spacelike coordinate σ. In this section we'll focus on closed strings and take σ to be periodic, with range

$$\sigma \in [0, 2\pi) \ . \tag{1.11}$$

We will sometimes package the two worldsheet coordinates together as $\sigma^\alpha = (\tau, \sigma)$, $\alpha = 0, 1$. Then the string sweeps out a surface in spacetime which defines a map from the worldsheet to Minkowski space, $X^\mu(\sigma, \tau)$ with $\mu = 0, \ldots, D - 1$. For closed strings, we require

Figure 5:

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau) \ .$$

In this context, spacetime is sometimes referred to as the *target space* to distinguish it from the worldsheet.

We need an action that describes the dynamics of this string. The key property that we will ask for is that nothing depends on the coordinates σ^α that we choose on the worldsheet. In other words, the string action should be reparameterization invariant. What kind of action does the trick? Well, for the point particle the action was proportional to the length of the worldline. The obvious generalization is that the action for the string should be proportional to the area, A, of the worldsheet. This is certainly a property that is characteristic of the worldsheet itself, rather than any choice of parameterization.

How do we find the area A in terms of the coordinates $X^\mu(\sigma, \tau)$? The worldsheet is a curved surface embedded in spacetime. The induced metric, $\gamma_{\alpha\beta}$, on this surface is the pull-back of the flat metric on Minkowski space,

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \ . \tag{1.12}$$

Then the action which is proportional to the area of the worldsheet is given by,

$$S = -T \int d^2\sigma \ \sqrt{-\det \gamma} \ . \tag{1.13}$$

Here T is a constant of proportionality. We will see shortly that it is the *tension* of the string, meaning the mass per unit length.
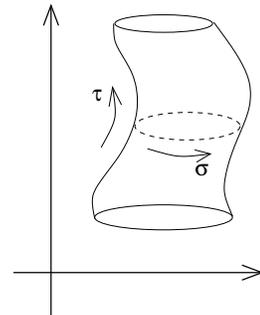
We can write this action a little more explicitly. The pull-back of the metric is given by,

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & X'^2 \end{pmatrix} .$$

where $\dot{X}^\mu = \partial X^\mu / \partial\tau$ and $X^{\mu\prime} = \partial X^\mu / \partial\sigma$. The action then takes the form,

$$S = -T \int d^2\sigma \sqrt{-(\dot{X})^2 \, (X')^2 + (\dot{X} \cdot X')^2} \, . \tag{1.14}$$

This is the *Nambu-Goto* action for a relativistic string.

Action = Area: A Check

If you're unfamiliar with differential geometry, the argument about the pull-back of the metric may be a bit slick. Thankfully, there's a more pedestrian way to see that the action (1.14) is equal to the area swept out by the world-sheet. It's slightly simpler to make this argument for a surface embedded in Euclidean space rather than Minkowski space. We choose some parameterization of the sheet in terms of τ and σ, as drawn in the figure, and we write the

Figure 6:

coordinates of Euclidean space as $\vec{X}(\sigma, \tau)$. We'll compute the area of the infinitesimal shaded region. The vectors tangent to the boundary are,
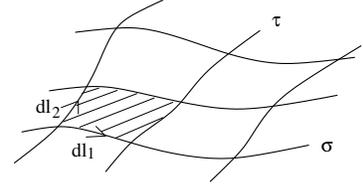
$$\vec{dl}_1 = \frac{\partial \vec{X}}{\partial \sigma} \quad , \quad \vec{dl}_2 = \frac{\partial \vec{X}}{\partial \tau} \, .$$

If the angle between these two vectors is θ, then the area is then given by

$$ds^2 = |\vec{dl}_1||\vec{dl}_2| \sin\theta = \sqrt{dl_1^2 \, dl_2^2 (1 - \cos^2\theta)} = \sqrt{dl_1^2 \, dl_2^2 - (\vec{dl}_1 \cdot \vec{dl}_2)^2} \tag{1.15}$$

which indeed takes the form of the integrand of (1.14).

Tension and Dimension

Let's now see that T has the physical interpretation of tension. We write Minkowski coordinates as $X^\mu = (t, \vec{x})$. We work in a gauge with $X^0 \equiv t = R\tau$, where R is a constant that is needed to balance up dimensions (see below) and will drop out at the end of the argument. Consider a snapshot of a string configuration at a time when

$d\vec{x}/d\tau = 0$ so that the instantaneous kinetic energy vanishes. Evaluating the action for a time dt gives

$$S = -T \int d\tau d\sigma R \sqrt{(d\vec{x}/d\sigma)^2} = -T \int dt \, (\text{spatial length of string}) \,. \qquad (1.16)$$

But, when the kinetic energy vanishes, the action is proportional to the time integral of the potential energy,

$$\text{potential energy} = T \times (\text{spatial length of string}) \,.$$

So T is indeed the energy per unit length as claimed. We learn that the string acts rather like an elastic band and its energy increases linearly with length. (This is different from the elastic bands you're used to which obey Hooke's law where energy increased quadratically with length). To minimize its potential energy, the string will want to shrink to zero size. We'll see that when we include quantum effects this can't happen because of the usual zero point energies.

There is a slightly annoying way of writing the tension that has its origin in ancient history, but is commonly used today

$$T = \frac{1}{2\pi\alpha'} \qquad (1.17)$$

where α' is pronounced "alpha-prime". In the language of our ancestors, α' is referred to as the "universal Regge slope". We'll explain why later in this course.

At this point, it's worth pointing out some conventions that we have, until now, left implicit. The spacetime coordinates have dimension $[X] = -1$. In contrast, the worldsheet coordinates are taken to be dimensionless, $[\sigma] = 0$. (This can be seen in our identification $\sigma \equiv \sigma + 2\pi$). The tension is equal to the mass per unit length and has dimension $[T] = 2$. Obviously this means that $[\alpha'] = -2$. We can therefore associate a length scale, l_s, by

$$\alpha' = l_s^2 \qquad (1.18)$$

The *string scale* l_s is the natural length that appears in string theory. In fact, in a certain sense (that we will make more precise later in the course) this length scale is the only parameter of the theory.

Actual Strings vs. Fundamental Strings

There are several situations in Nature where string-like objects arise. Prime examples include magnetic flux tubes in superconductors and chromo-electric flux tubes in QCD. Cosmic strings, a popular speculation in cosmology, are similar objects, stretched across the sky. In each of these situations, there are typically two length scales associated to the string: the tension, T and the width of the string, L. For all these objects, the dynamics is governed by the Nambu-Goto action as long as the curvature of the string is much greater than L. (In the case of superconductors, one should work with a suitable non-relativistic version of the Nambu-Goto action).

However, in each of these other cases, the Nambu-Goto action is not the end of the story. There will typically be additional terms in the action that depend on the width of the string. The form of these terms is not universal, but often includes a *rigidity* piece of form $L \int K^2$, where K is the extrinsic curvature of the worldsheet. Other terms could be added to describe fluctuations in the width of the string.

The string scale, l_s, or equivalently the tension, T, depends on the kind of string that we're considering. For example, if we're interested in QCD flux tubes then we would take

$$T \sim (1 \text{ GeV})^2 \tag{1.19}$$

In this course we will consider *fundamental strings* which have zero width. What this means in practice is that we take the Nambu-Goto action as the complete description for all configurations of the string. These strings will have relevance to quantum gravity and the tension of the string is taken to be much larger, typically an order of magnitude or so below the Planck scale.

$$T \lesssim M_{pl}^2 = (10^{18} \text{ GeV})^2 \tag{1.20}$$

However, I should point out that when we try to view string theory as a fundamental theory of quantum gravity, we don't really know what value T should take. As we will see later in this course, it depends on many other aspects, most notably the string coupling and the volume of the extra dimensions.

1.2.1 Symmetries of the Nambu-Goto Action

The Nambu-Goto action has two types of symmetry, each of a different nature.

- Poincaré invariance of the spacetime (1.6). This is a global symmetry from the perspective of the worldsheet, meaning that the parameters $\Lambda^\mu{}_\nu$ and c^μ which label

the symmetry transformation are constants and do not depend on worldsheet coordinates σ^α.

- Reparameterization invariance, $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$. As for the point particle, this is a gauge symmetry. It reflects the fact that we have a redundancy in our description because the worldsheet coordinates σ^α have no physical meaning.

1.2.2 Equations of Motion

To derive the equations of motion for the Nambu-Goto string, we first introduce the momenta which we call Π because there will be countless other quantities that we want to call p later,

$$\Pi^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T \frac{(\dot{X} \cdot X')X'_\mu - (X'^2)\dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 \, X'^2}} \;.$$

$$\Pi^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X'^\mu} = -T \frac{(\dot{X} \cdot X')\dot{X}_\mu - (\dot{X}^2)X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 \, X'^2}} \;.$$

The equations of motion are then given by,

$$\frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \sigma} = 0$$

These look like nasty, non-linear equations. In fact, there's a slightly nicer way to write these equations, starting from the earlier action (1.13). Recall that the variation of a determinant is $\delta\sqrt{-\gamma} = \frac{1}{2}\sqrt{-\gamma}\,\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta}$. Using the definition of the pull-back metric $\gamma_{\alpha\beta}$, this gives rise to the equations of motion

$$\partial_\alpha(\sqrt{-\det\gamma}\,\gamma^{\alpha\beta}\partial_\beta X^\mu) = 0 \;, \tag{1.21}$$

Although this notation makes the equations look a little nicer, we're kidding ourselves. Written in terms of X^μ, they are still the same equations. Still nasty.

1.3 The Polyakov Action

The square-root in the Nambu-Goto action means that it's rather difficult to quantize using path integral techniques. However, there is another form of the string action which is classically equivalent to the Nambu-Goto action. It eliminates the square root at the expense of introducing another field,

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \, \eta_{\mu\nu} \tag{1.22}$$

where $g \equiv \det g$. This is the *Polyakov* action. (Polyakov didn't discover the action, but he understood how to work with it in the path integral and for this reason it carries his name. The path integral treatment of this action will be the subject of Chapter 5).

The new field is $g_{\alpha\beta}$. It is a dynamical metric on the worldsheet. From the perspective of the worldsheet, the Polyakov action is a bunch of scalar fields X coupled to 2d gravity.

The equation of motion for X^μ is

$$\partial_\alpha(\sqrt{-g}\,g^{\alpha\beta}\partial_\beta X^\mu) = 0 \ , \tag{1.23}$$

which coincides with the equation of motion (1.21) from the Nambu-Goto action, except that $g_{\alpha\beta}$ is now an independent variable which is fixed by its own equation of motion. To determine this, we vary the action (remembering again that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\alpha\beta}\delta g^{\alpha\beta} = +\frac{1}{2}\sqrt{-g}\,g^{\alpha\beta}\delta g_{\alpha\beta}$),

$$\delta S = -\frac{T}{2}\int d^2\sigma \ \delta g^{\alpha\beta}\left(\sqrt{-g}\,\partial_\alpha X^\mu \partial_\beta X^\nu - \tfrac{1}{2}\sqrt{-g}\,g_{\alpha\beta}g^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X^\nu\right)\eta_{\mu\nu} = 0 \ . \tag{1.24}$$

The worldsheet metric is therefore given by,

$$g_{\alpha\beta} = 2f(\sigma)\,\partial_\alpha X \cdot \partial_\beta X \ , \tag{1.25}$$

where the function $f(\sigma)$ is given by,

$$f^{-1} = g^{\rho\sigma}\,\partial_\rho X \cdot \partial_\sigma X$$

A comment on the potentially ambiguous notation: here, and below, any function $f(\sigma)$ is always short-hand for $f(\sigma,\tau)$: it in no way implies that f depends only on the spatial worldsheet coordinate.

We see that $g_{\alpha\beta}$ isn't quite the same as the pull-back metric $\gamma_{\alpha\beta}$ defined in equation (1.12); the two differ by the conformal factor f. However, this doesn't matter because, rather remarkably, f drops out of the equation of motion (1.23). This is because the $\sqrt{-g}$ term scales as f, while the inverse metric $g^{\alpha\beta}$ scales as f^{-1} and the two pieces cancel. We therefore see that Nambu-Goto and the Polyakov actions result in the same equation of motion for X.

In fact, we can see more directly that the Nambu-Goto and Polyakov actions coincide. We may replace $g_{\alpha\beta}$ in the Polyakov action (1.22) with its equation of motion $g_{\alpha\beta} = 2f\,\gamma_{\alpha\beta}$. The factor of f also drops out of the action for the same reason that it dropped out of the equation of motion. In this manner, we recover the Nambu-Goto action (1.13).

1.3.1 Symmetries of the Polyakov Action

The fact that the presence of the factor $f(\sigma, \tau)$ in (1.25) didn't actually affect the equations of motion for X^μ reflects the existence of an extra symmetry which the Polyakov action enjoys. Let's look more closely at this. Firstly, the Polyakov action still has the two symmetries of the Nambu-Goto action,

- Poincaré invariance. This is a global symmetry on the worldsheet.

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \ .$$

- Reparameterization invariance, also known as diffeomorphisms. This is a gauge symmetry on the worldsheet. We may redefine the worldsheet coordinates as $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$. The fields X^μ transform as worldsheet scalars, while $g_{\alpha\beta}$ transforms in the manner appropriate for a 2d metric.

$$X^\mu(\sigma) \to \tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma)$$
$$g_{\alpha\beta}(\sigma) \to \tilde{g}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial \sigma^\gamma}{\partial \tilde{\sigma}^\alpha} \frac{\partial \sigma^\delta}{\partial \tilde{\sigma}^\beta} g_{\gamma\delta}(\sigma)$$

It will sometimes be useful to work infinitesimally. If we make the coordinate change $\sigma^\alpha \to \tilde{\sigma}^\alpha = \sigma^\alpha - \eta^\alpha(\sigma)$, for some small η. The transformations of the fields then become,

$$\delta X^\mu(\sigma) = \eta^\alpha \partial_\alpha X^\mu$$
$$\delta g_{\alpha\beta}(\sigma) = \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha$$

where the covariant derivative is defined by $\nabla_\alpha \eta_\beta = \partial_\alpha \eta_\beta - \Gamma^\sigma_{\alpha\beta} \eta_\sigma$ with the Levi-Civita connection associated to the worldsheet metric given by the usual expression,

$$\Gamma^\sigma_{\alpha\beta} = \tfrac{1}{2} g^{\sigma\rho}(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta})$$

Together with these familiar symmetries, there is also a new symmetry which is novel to the Polyakov action. It is called *Weyl invariance*.

- Weyl Invariance. Under this symmetry, $X^\mu(\sigma) \to X^\mu(\sigma)$, while the metric changes as

$$g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma)\, g_{\alpha\beta}(\sigma) \ . \tag{1.26}$$

Or, infinitesimally, we can write $\Omega^2(\sigma) = e^{2\phi(\sigma)}$ for small ϕ so that

$$\delta g_{\alpha\beta}(\sigma) = 2\phi(\sigma)\, g_{\alpha\beta}(\sigma) \ .$$

It is simple to see that the Polyakov action is invariant under this transformation: the factor of Ω^2 drops out just as the factor of f did in equation (1.25), canceling between $\sqrt{-g}$ and the inverse metric $g^{\alpha\beta}$. This is a gauge symmetry of the string, as seen by the fact that the parameter Ω depends on the worldsheet coordinates σ. This means that two metrics which are related by a Weyl transformation (1.26) are to be considered as the same physical state.
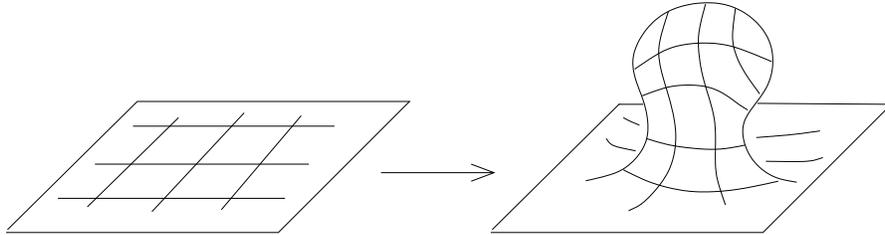

Figure 7: An example of a Weyl transformation

How should we think of Weyl invariance? It is not a coordinate change. Instead it is the invariance of the theory under a local change of scale which preserves the angles between all lines. For example the two worldsheet metrics shown in the figure are viewed by the Polyakov string as equivalent. This is rather surprising! And, as you might imagine, theories with this property are extremely rare. It should be clear from the discussion above that the property of Weyl invariance is special to two dimensions, for only there does the scaling factor coming from the determinant $\sqrt{-g}$ cancel that coming from the inverse metric. But even in two dimensions, if we wish to keep Weyl invariance then we are strictly limited in the kind of interactions that can be added to the action. For example, we would not be allowed a potential term for the worldsheet scalars of the form,

$$\int d^2\sigma \sqrt{-g}\, V(X)\ .$$

These break Weyl invariance. Nor can we add a worldsheet cosmological constant term,

$$\mu \int d^2\sigma \sqrt{-g}\ .$$

This too breaks Weyl invariance. We will see later in this course that the requirement of Weyl invariance becomes even more stringent in the quantum theory. We will also see what kind of interactions terms can be added to the worldsheet. Indeed, much of this course can be thought of as the study of theories with Weyl invariance.

1.3.2 Fixing a Gauge

As we have seen, the equation of motion (1.23) looks pretty nasty. However, we can use the redundancy inherent in the gauge symmetry to choose coordinates in which they simplify. Let's think about what we can do with the gauge symmetry.

Firstly, we have two reparameterizations to play with. The worldsheet metric has three independent components. This means that we expect to be able to set any two of the metric components to a value of our choosing. We will choose to make the metric locally conformally flat, meaning

$$g_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta} \ , \tag{1.27}$$

where $\phi(\sigma, \tau)$ is some function on the worldsheet. You can check that this is possible by writing down the change of the metric under a coordinate transformation and seeing that the differential equations which result from the condition (1.27) have solutions, at least locally. Choosing a metric of the form (1.27) is known as *conformal gauge*.

We have only used reparameterization invariance to get to the metric (1.27). We still have Weyl transformations to play with. Clearly, we can use these to remove the last independent component of the metric and set $\phi = 0$ such that,

$$g_{\alpha\beta} = \eta_{\alpha\beta} \ . \tag{1.28}$$

We end up with the flat metric on the worldsheet in Minkowski coordinates.

A Diversion: How to make a metric flat

The fact that we can use Weyl invariance to make any two-dimensional metric flat is an important result. Let's take a quick diversion from our main discussion to see a different proof that isn't tied to the choice of Minkowski coordinates on the worldsheet. We'll work in 2d Euclidean space to avoid annoying minus signs. Consider two metrics related by a Weyl transformation, $g'_{\alpha\beta} = e^{2\phi}g_{\alpha\beta}$. One can check that the Ricci scalars of the two metrics are related by,

$$\sqrt{g'}R' = \sqrt{g}(R - 2\nabla^2\phi) \ . \tag{1.29}$$

We can therefore pick a ϕ such that the new metric has vanishing Ricci scalar, $R' = 0$, simply by solving this differential equation for ϕ. However, in two dimensions (but not in higher dimensions) a vanishing Ricci scalar implies a flat metric. The reason is simply that there aren't too many indices to play with. In particular, symmetry of the Riemann tensor in two dimensions means that it must take the form,

$$R_{\alpha\beta\gamma\delta} = \frac{R}{2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \ .$$

So $R' = 0$ is enough to ensure that $R'_{\alpha\beta\gamma\delta} = 0$, which means that the manifold is flat. In equation (1.28), we've further used reparameterization invariance to pick coordinates in which the flat metric is the Minkowski metric.

The equations of motion and the stress-energy tensor

With the choice of the flat metric (1.28), the Polyakov action simplifies tremendously and becomes the theory of D free scalar fields. (In fact, this simplification happens in any conformal gauge).

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \ \partial_\alpha X \cdot \partial^\alpha X \ , \tag{1.30}$$

and the equations of motion for X^μ reduce to the free wave equation,

$$\partial_\alpha \partial^\alpha X^\mu = 0 \ . \tag{1.31}$$

Now that looks too good to be true! Are the horrible equations (1.23) really equivalent to a free wave equation? Well, not quite. There is something that we've forgotten: we picked a choice of gauge for the metric $g_{\alpha\beta}$. But we must still make sure that the equation of motion for $g_{\alpha\beta}$ is satisfied. In fact, the variation of the action with respect to the metric gives rise to a rather special quantity: it is the stress-energy tensor, $T_{\alpha\beta}$. With a particular choice of normalization convention, we define the stress-energy tensor to be

$$T_{\alpha\beta} = -\frac{2}{T} \frac{1}{\sqrt{-g}} \frac{\partial S}{\partial g^{\alpha\beta}} \ .$$

We varied the Polyakov action with respect to $g_{\alpha\beta}$ in (1.24). When we set $g_{\alpha\beta} = \eta_{\alpha\beta}$ we get

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \tfrac{1}{2}\eta_{\alpha\beta} \, \eta^{\rho\sigma} \partial_\rho X \cdot \partial_\sigma X \ . \tag{1.32}$$

The equation of motion associated to the metric $g_{\alpha\beta}$ is simply $T_{\alpha\beta} = 0$. Or, more explicitly,

$$T_{01} = \dot{X} \cdot X' = 0$$
$$T_{00} = T_{11} = \tfrac{1}{2}(\dot{X}^2 + X'^2) = 0 \ . \tag{1.33}$$

We therefore learn that the equations of motion of the string are the free wave equations (1.31) subject to the two constraints (1.33) arising from the equation of motion $T_{\alpha\beta} = 0$.

Getting a feel for the constraints

Let's try to get some intuition for these constraints. There is a simple meaning of the first constraint in (1.33): we must choose our parameterization such that lines of constant σ are perpendicular to the lines of constant τ, as shown in the figure.
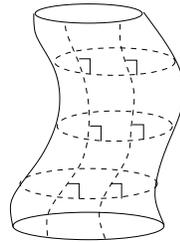
But we can do better. To gain more physical insight, we need to make use of the fact that we haven't quite exhausted our gauge symmetry. We will discuss this more in Section 2.2, but for now one can check that there is enough remnant gauge symmetry to allow us to go to static gauge,

Figure 8:

$$X^0 \equiv t = R\tau \ ,$$

so that $(X^0)' = 0$ and $\dot{X}^0 = R$, where R is a constant that is needed on dimensional grounds. The interpretation of this constant will become clear shortly. Then, writing $X^\mu = (t, \vec{x})$, the equation of motion for spatial components is the free wave equation,

$$\ddot{\vec{x}} - \vec{x}'' = 0$$

while the constraints become

$$\dot{\vec{x}} \cdot \vec{x}' = 0$$
$$\dot{\vec{x}}^{\,2} + \vec{x}'^{\,2} = R^2 \tag{1.34}$$

The first constraint tells us that the motion of the string must be perpendicular to the string itself. In other words, the physical modes of the string are transverse oscillations. There is no longitudinal mode. We'll also see this again in Section 2.2.

From the second constraint, we can understand the meaning of the constant R: it is related to the length of the string when $\dot{\vec{x}} = 0$,

$$\int d\sigma \ \sqrt{(d\vec{x}/d\sigma)^2} = 2\pi R \ .$$

Of course, if we have a stretched string with $\dot{\vec{x}} = 0$ at one moment of time, then it won't stay like that for long. It will contract under its own tension. As this happens, the second constraint equation relates the length of the string to the instantaneous velocity of the string.

1.4 Mode Expansions

Let's look at the equations of motion and constraints more closely. The equations of motion (1.31) are easily solved. We introduce lightcone coordinates on the worldsheet,

$$\sigma^{\pm} = \tau \pm \sigma \ ,$$

in terms of which the equations of motion simply read

$$\partial_+ \partial_- X^{\mu} = 0$$

The most general solution is,

$$X^{\mu}(\sigma, \tau) = X_L^{\mu}(\sigma^+) + X_R^{\mu}(\sigma^-)$$

for arbitrary functions X_L^{μ} and X_R^{μ}. These describe left-moving and right-moving waves respectively. Of course the solution must still obey both the constraints (1.33) as well as the periodicity condition,

$$X^{\mu}(\sigma, \tau) = X^{\mu}(\sigma + 2\pi, \tau) \ . \tag{1.35}$$

The most general, periodic solution can be expanded in Fourier modes,

$$X_L^{\mu}(\sigma^+) = \tfrac{1}{2} x^{\mu} + \tfrac{1}{2} \alpha' p^{\mu} \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^{\mu} e^{-in\sigma^+} \ ,$$

$$X_R^{\mu}(\sigma^-) = \tfrac{1}{2} x^{\mu} + \tfrac{1}{2} \alpha' p^{\mu} \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^{\mu} e^{-in\sigma^-} \ . \tag{1.36}$$

This mode expansion will be very important when we come to the quantum theory. Let's make a few simple comments here.

- Various normalizations in this expression, such as the α' and factor of $1/n$ have been chosen for later convenience.

- X_L and X_R do not individually satisfy the periodicity condition (1.35) due to the terms linear in $\sigma^{\pm}$. However, the sum of them is invariant under $\sigma \to \sigma + 2\pi$ as required.

- The variables x^{μ} and p^{μ} are the position and momentum of the center of mass of the string. This can be checked, for example, by studying the Noether currents arising from the spacetime translation symmetry $X^{\mu} \to X^{\mu} + c^{\mu}$. One finds that the conserved charge is indeed p^{μ}.

- Reality of X^{μ} requires that the coefficients of the Fourier modes, α_n^{μ} and $\tilde{\alpha}_n^{\mu}$, obey

$$\alpha_n^{\mu} = (\alpha_{-n}^{\mu})^{\star} \quad , \quad \tilde{\alpha}_n^{\mu} = (\tilde{\alpha}_{-n}^{\mu})^{\star} \ . \tag{1.37}$$

1.4.1 The Constraints Revisited

We still have to impose the two constraints (1.33). In the worldsheet lightcone coordinates $\sigma^{\pm}$, these become,

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0 \ . \tag{1.38}$$

These equations give constraints on the momenta p^{μ} and the Fourier modes α_n^{μ} and $\tilde{\alpha}_n^{\mu}$. To see what these are, let's look at

$$\partial_- X^{\mu} = \partial_- X_R^{\mu} = \frac{\alpha'}{2} p^{\mu} + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^{\mu} e^{-in\sigma^-}$$

$$= \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^{\mu} e^{-in\sigma^-}$$

where in the second line the sum is over all $n \in \mathbf{Z}$ and we have defined α_0^{μ} to be

$$\alpha_0^{\mu} \equiv \sqrt{\frac{\alpha'}{2}} p^{\mu} \ .$$

The constraint (1.38) can then be written as

$$(\partial_- X)^2 = \frac{\alpha'}{2} \sum_{m,p} \alpha_m \cdot \alpha_p \, e^{-i(m+p)\sigma^-}$$

$$= \frac{\alpha'}{2} \sum_{m,n} \alpha_m \cdot \alpha_{n-m} \, e^{-in\sigma^-}$$

$$\equiv \alpha' \sum_n L_n \, e^{-in\sigma^-} = 0 \ .$$

where we have defined the sum of oscillator modes,

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m \ . \tag{1.39}$$

We can also do the same for the left-moving modes, where we again define an analogous sum of operator modes,

$$\tilde{L}_n = \frac{1}{2} \sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m \ . \tag{1.40}$$

with the zero mode defined to be,

$$\tilde{\alpha}_0^{\mu} \equiv \sqrt{\frac{\alpha'}{2}} p^{\mu} \ .$$

The fact that $\tilde{\alpha}_0^\mu = \alpha_0^\mu$ looks innocuous but is a key point to remember when we come to quantize the string. The L_n and $\tilde{L}_n$ are the Fourier modes of the constraints. Any classical solution of the string of the form (1.36) must further obey the infinite number of constraints,

$$L_n = \tilde{L}_n = 0 \quad n \in \mathbf{Z} \ .$$

We'll meet these objects L_n and $\tilde{L}_n$ again in a more general context when we come to discuss conformal field theory.

The constraints arising from L_0 and $\tilde{L}_0$ have a rather special interpretation. This is because they include the square of the spacetime momentum p^μ. But, the square of the spacetime momentum is an important quantity in Minkowski space: it is the square of the rest mass of a particle,

$$p_\mu p^\mu = -M^2 \ .$$

So the L_0 and $\tilde{L}_0$ constraints tell us the effective mass of a string in terms of the excited oscillator modes, namely

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n} \tag{1.41}$$

Because both α_0^μ and $\tilde{\alpha}_0^\mu$ are equal to $\sqrt{\alpha'/2}\, p^\mu$, we have two expressions for the invariant mass: one in terms of right-moving oscillators α_n^μ and one in terms of left-moving oscillators $\tilde{\alpha}_n^\mu$. And these two terms must be equal to each other. This is known as *level matching*. It will play an important role in the next section where we turn to the quantum theory.