8. Compactification and T-Duality

In this section, we will consider the simplest compactification of the bosonic string: a background spacetime of the form

$$\mathbf{R}^{1,24} \times \mathbf{S}^1 \tag{8.1}$$

The circle is taken to have radius R, so that the coordinate on $\mathbf{S}^1$ has periodicity

$$X^{25} \equiv X^{25} + 2\pi R$$

We will initially be interested in the physics at length scales $\gg R$ where motion on the $\mathbf{S}^1$ can be ignored. Our goal is to understand what physics looks like to an observer living in the non-compact $\mathbf{R}^{1,24}$ Minkowski space. This general idea goes by the name of *Kaluza-Klein compactification*. We will view this compactification in two ways: firstly from the perspective of the spacetime low-energy effective action and secondly from the perspective of the string worldsheet.

8.1 The View from Spacetime

Let's start with the low-energy effective action. Looking at length scales $\gg R$ means that we will take all fields to be independent of X^{25}: they are instead functions only on the non-compact $\mathbf{R}^{1,24}$.

Consider the metric in Einstein frame. This decomposes into three different fields on $\mathbf{R}^{1,24}$: a metric $\tilde{G}_{\mu\nu}$, a vector A_μ and a scalar σ which we package into the $D = 26$ dimensional metric as

$$ds^2 = \tilde{G}_{\mu\nu}\, dX^\mu\, dX^\nu + e^{2\sigma}\left(dX^{25} + A_\mu\, dX^\mu\right)^2 \tag{8.2}$$

Here all the indices run over the non-compact directions $\mu, \nu = 0, \ldots 24$ only.

The vector field A_μ is an honest gauge field, with the gauge symmetry descending from diffeomorphisms in $D = 26$ dimensions. To see this recall that under the transformation $\delta X^\mu = V^\mu(X)$, the metric transforms as

$$\delta G_{\mu\nu} = \nabla_\mu \Lambda_\nu + \nabla_\nu \Lambda_\mu$$

This means that diffeomorphisms of the compact direction, $\delta X^{25} = \Lambda(X^\mu)$, turn into gauge transformations of A_μ,

$$\delta A_\mu = \partial_\mu \Lambda$$

We'd like to know how the fields $G_{\mu\nu}$, A_μ and σ interact. To determine this, we simply insert the ansatz (8.2) into the $D = 26$ Einstein-Hilbert action. The $D = 26$ Ricci scalar $\mathcal{R}^{(26)}$ is given by

$$\mathcal{R}^{(26)} = \mathcal{R} - 2e^{-\sigma}\nabla^2 e^{\sigma} - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}$$

where $\mathcal{R}$ in this formula now refers to the $D = 25$ Ricci scalar. The action governing the dynamics becomes

$$S = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-\tilde{G}^{(26)}}\;\mathcal{R}^{(26)} = \frac{2\pi R}{2\kappa^2}\int d^{25}X\sqrt{-\tilde{G}}\;e^{\sigma}\left(\mathcal{R} - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\sigma\partial^\mu\sigma\right)$$

The dimensional reduction of Einstein gravity in D dimensions gives Einstein gravity in $D-1$ dimensions, coupled to a $U(1)$ gauge theory and a single massless scalar. This illustrates the original idea of Kaluza and Klein, with Maxwell theory arising naturally from higher-dimensional gravity.

The gravitational action above is not quite of the Einstein-Hilbert form. We need to again change frames, absorbing the scalar σ in the same manner as we absorbed the dilaton in Section 7.3.1. Moreover, just as for the dilaton, there is no potential dictating the vacuum expectation value of σ. Changing the vev of σ corresponds to changing R, so this is telling us that nothing in the gravitational action fixes the radius R of the compact circle. This is a problem common to all Kaluza-Klein compactifications[11]: there are always massless scalar fields, corresponding to the volume of the internal space as well as other deformations. Massless scalar fields, such as the dilaton Φ or the volume σ, are usually referred to as *moduli*.

If we want this type of Kaluza-Klein compactification to describe our universe — where we don't see massless scalar fields — we need to find a way to "fix the moduli". This means that we need a mechanism which gives rise to a potential for the scalar fields, making them heavy and dynamically fixing their vacuum expectation value. Such mechanisms exist in the context of the superstring.

Let's now also look at the Kaluza-Klein reduction of the other fields in the low-energy effective action. The dilaton is easy: a scalar in D dimensions reduces to a scalar in $D-1$ dimensions. The anti-symmetric 2-form has more structure: it reduces to a 2-form $B_{\mu\nu}$, together with a vector field $\tilde{A}_\mu = B_{\mu\,25}$.

[11]The description of compactification on more general manifolds is a beautiful story involving aspects differential geometry and topology. This story is told in the second volume of Green, Schwarz and Witten.

In summary, the low-energy physics of the bosonic string in $D-1$ dimensions consists of a metric $G_{\mu\nu}$, two $U(1)$ gauge fields A_μ and $\tilde{A}_\mu$ and two massless scalars Φ and σ.

8.1.1 Moving around the Circle

In the above discussion, we assumed that all fields are independent of the periodic direction X^{25}. Let's now look at what happens if we relax this constraint. It's simplest to see the resulting physics if we look at the scalar field Φ where we don't have to worry about cluttering equations with indices. In general, we can expand this field in Fourier modes around the circle

$$\Phi(X^\mu; X^{25}) = \sum_{n=-\infty}^{\infty} \Phi_n(X^\mu) e^{inX^{25}/R}$$

where reality requires $\Phi_n^\star = \Phi_{-n}$. Ignoring the coupling to gravity for now, the kinetic terms for this scalar are

$$\int d^{26}X\ \partial_\mu \Phi\, \partial^\mu \Phi + (\partial_{25}\Phi)^2 = 2\pi R \int d^{25}X\ \sum_{n=-\infty}^{\infty} \left(\partial_\mu \Phi_n\, \partial^\mu \Phi_{-n} + \frac{n^2}{R^2}\,|\,\Phi_n|^2 \right)$$

This simple Fourier decomposition is telling us something very important: a single scalar field on $\mathbf{R}^{1,D-1} \times \mathbf{S}^1$ splits into an infinite number of scalar fields on $\mathbf{R}^{1,D-2}$, indexed by the integer n. These have mass

$$M_n^2 = \frac{n^2}{R^2} \tag{8.3}$$

For R small, all particles are heavy except for the massless zero mode $n = 0$. The heavy particles are typically called Kaluza-Klein (KK) modes and can be ignored if we're probing energies $\ll 1/R$ or, equivalently, distance scales $\gg R$.

There is one further interesting property of the KK modes Φ_n with $n \neq 0$: they are charged under the gauge field A_μ arising from the metric. The simplest way to see this is to look at the appropriate gauge transformation which, from the spacetime perspective, is the diffeomorphism $X^{25} \to X^{25} + \Lambda(X^\mu)$. Clearly, this shifts the KK modes

$$\Phi_n \to \exp\left(\frac{in\Lambda}{R}\right) \Phi_n$$

This tells us that the n^{th} KK mode has charge n/R. In fact, one usually rescales the gauge field to $A'_\mu = A_\mu/R$, under which the charge of the KK mode Φ_n is simply $n \in \mathbf{Z}$.

8.2 The View from the Worldsheet

We now consider the Kaluza-Klein reduction from the perspective of the string. We want to study a string moving in the background $\mathbf{R}^{1,24} \times \mathbf{S}^1$. There are two ways in which the compact circle changes the string dynamics.

The first effect of the circle is that the spatial momentum, p, of the string in the circle direction can no longer take any value, but is quantized in integer units

$$p^{25} = \frac{n}{R} \qquad n \in \mathbf{Z}$$

The simplest way to see this is simply to require that the string wavefunction, which includes the factor $e^{ip\cdot X}$, is single valued.

The second effect is that we can allow more general boundary conditions for the mode expansion of X. As we move around the string, we no longer need $X(\sigma + 2\pi) = X(\sigma)$, but can relax this to

$$X^{25}(\sigma + 2\pi) = X^{25}(\sigma) + 2\pi m R \qquad m \in \mathbf{Z}$$

The integer m tells us how many times the string winds around $\mathbf{S}^1$. It is usually simply called the *winding number*.

Let's now follow the familiar path that we described in Section 2 to study the spectrum of the string on the spacetime (8.1). We start by considering only the periodic field X^{25}, highlighting the differences with our previous treatment. The mode expansion of X^{25} is now given by

$$X^{25}(\sigma, \tau) = x^{25} + \frac{\alpha' n}{R}\tau + mR\sigma + \text{oscillator modes}$$

which incorporates both the quantized momentum and the possibility of a winding number. Before splitting $X^{25}(\sigma, \tau)$ into right-moving and left-moving parts, it will be useful to introduce the quantities

$$p_L = \frac{n}{R} + \frac{mR}{\alpha'} \quad , \quad p_R = \frac{n}{R} - \frac{mR}{\alpha'} \tag{8.4}$$

Then we have $X^{25}(\sigma, \tau) = X^{25}_L(\sigma^+) + X^{25}_R(\sigma^-)$, where

$$X^{25}_L(\sigma^+) = \tfrac{1}{2}x^{25} + \tfrac{1}{2}\alpha' p_L\, \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\, \tilde{\alpha}^{25}_n\, e^{-in\sigma^+} \ ,$$

$$X^{25}_R(\sigma^-) = \tfrac{1}{2}x^{25} + \tfrac{1}{2}\alpha' p_R\, \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\, \alpha^{25}_n\, e^{-in\sigma^-}$$

This differs from the mode expansion (1.36) only in the terms p_L and p_R. The mode expansion for all the other scalar fields on flat space $\mathbf{R}^{1,24}$ remains unchanged and we don't write them explicitly.

Let's think about what the spectrum of this theory looks like to an observer living in $D = 25$ non-compact directions. Each particle state will be described by a momentum p^μ with $\mu = 0, \ldots, 24$. The mass of the particle is

$$M^2 = -\sum_{\mu=0}^{24} p_\mu p^\mu$$

As before, the mass of these particles is fixed in terms of the oscillator modes of the string by the L_0 and $\tilde{L}_0$ equations. These now read

$$M^2 = p_L^2 + \frac{4}{\alpha'}(\tilde{N} - 1) = p_R^2 + \frac{4}{\alpha'}(N - 1)$$

where N and $\tilde{N}$ are the levels, defined in lightcone quantization by (2.24). (One should take the lightcone coordinate inside $\mathbf{R}^{1,24}$ rather than along the $\mathbf{S}^1$). The factors of -1 are the necessary normal ordering coefficients that we've seen in several guises in this course.

These equations differ from (2.25) by the presence of the momentum and winding terms around $\mathbf{S}^1$ on the right-hand side. In particular, level matching no longer tells us that $N = \tilde{N}$, but instead

$$N - \tilde{N} = nm \tag{8.5}$$

Expanding out the mass formula, we have

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{8.6}$$

The new terms in this formula have a simple interpretation. The first term tells us that a string with $n > 0$ units of momentum around the circle gains a contribution to its mass of n/R. This agrees with the result (8.3) that we found from studying the KK reduction of the spacetime theory. The second term is even easier to understand: a string which winds $m > 0$ times around the circle picks up a contribution $2\pi mRT$ to its mass, where $T = 1/2\pi\alpha'$ is the tension of the string.

8.2.1 Massless States

We now restrict attention to the massless states in $\mathbf{R}^{1,24}$. This can be achieved in the mass formula (8.6) by looking at states with zero momentum $n = 0$ and zero winding $m = 0$, obeying the level matching condition $N = \tilde{N} = 1$. The possibilities are

- $\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}\,|0;p\rangle$: Under the $SO(1,24)$ Lorentz group, these states decompose into a metric $G_{\mu\nu}$, an anti-symmetric tensor $B_{\mu\nu}$ and a scalar Φ.

- $\alpha^\mu_{-1}\tilde{\alpha}^{25}_{-1}\,|0;p\rangle$ and $\alpha^{25}_{-1}\tilde{\alpha}^\mu_{-1}\,|0;p\rangle$: These are two vector fields. We can identify the sum of these $(\alpha^\mu_{-1}\tilde{\alpha}^{25}_{-1} + \alpha^{25}_{-1}\tilde{\alpha}^\mu_{-1})\,|0;p\rangle$ with the vector field A_μ coming from the metric and the difference $(\alpha^\mu_{-1}\tilde{\alpha}^{25}_{-1} - \alpha^{25}_{-1}\tilde{\alpha}^\mu_{-1})\,|0;p\rangle$ with the vector field $\tilde{A}_\mu$ coming from the anti-symmetric field.

- $\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}\,|0;p\rangle$: This is another scalar. It is identified with the scalar σ associated to the radius of $\mathbf{S}^1$.

We see that the massless spectrum of the string coincides with the massless spectrum associated with the Kaluza-Klein reduction of the previous section.

8.2.2 Charged Fields

One can also check that the KK modes with $n \neq 0$ have charge n under the gauge field A_μ. We can determine the charge of a state under a given $U(1)$ by computing the 3-point function in which two legs correspond to the state of interest, while the third is the appropriate photon. We have two photons, with vertex operators given by,

$$V_\pm(p) \sim \int d^2z\; \zeta_\mu(\partial X^\mu \bar{\partial}\bar{X}^{25} \pm \partial X^{25}\bar{\partial}\bar{X}^\mu)e^{ip\cdot X}$$

where $+$ corresponds to A_μ and $-$ to $\tilde{A}_\mu$ and we haven't been careful about the overall normalization. Meanwhile, any state can be assigned momentum n and winding m by dressing the operator with the factor $e^{ip_L X^{25}(z) + ip_R \bar{X}^{25}(\bar{z})}$. As always, it's simplest to work with the momentum and winding modes of the tachyon, whose vertex operators are of the form

$$V_{m,n}(p) \sim \int d^2z\; e^{ip\cdot X}\; e^{ip_L X^{25} + ip_R \bar{X}^{25}}$$

The charge of a state is the coefficient in front of the 3-point coupling of the field and the photon,

$$\langle V_\pm(p_1)V_{m,n}(p_2)V_{-m,-n}(p_3)\rangle \sim \delta^{25}(\sum_i p_i)\,\zeta_\mu(p_2^\mu - p_3^\mu)\,(p_L \pm p_R)$$

The first few factors are merely kinematical. The interesting information is in the last factor. It is telling us that under A_μ, fields have charge $p_L + p_R \sim n/R$. This is in agreement with the Kaluza-Klein analysis that we saw before. However, it's also telling us something new: under $\tilde{A}_\mu$, fields have charge $p_L - p_R \sim mR/\alpha'$. In other words, winding modes are charged under the gauge field that arises from the reduction of $B_{\mu\nu}$. This is not surprising: winding modes correspond to strings wrapping the circle and we saw in Section 7 that strings are electrically charged under $B_{\mu\nu}$.

8.2.3 Enhanced Gauge Symmetry

With a circle in the game, there are other ways to build massless states that don't require us to work at level $N = \tilde{N} = 1$. For example, we can set $N = \tilde{N} = 0$ and look at winding modes $m \neq 0$. The level matching condition (8.5) requires $n = 0$ and the mass of the states is

$$M^2 = \left(\frac{mR}{\alpha'}\right)^2 - \frac{4}{\alpha'}$$

and states can be massless whenever the radius takes special values $R^2 = 4\alpha'/m^2$ with $m \in \mathbf{Z}$. Similarly, we can set the winding to zero $m = 0$ and consider the KK modes of the tachyon which have mass

$$M^2 = \frac{n^2}{R^2} - \frac{4}{\alpha'}$$

which become massless when $R^2 = n^2\alpha'/4$.

However, the richest spectrum of massless states occurs when the radius takes a very special value, namely

$$R = \sqrt{\alpha'}$$

Solutions to the level matching condition (8.5) with $M^2 = 0$ are now given by

- $N = \tilde{N} = 1$ with $m = n = 0$. These give the states described above: a metric, two $U(1)$ gauge fields and two neutral scalars.
- $N = \tilde{N} = 0$ with $n = \pm 2$ and $m = 0$. These are KK modes of the tachyon field. They are scalars in spacetime with charges $(\pm 2, 0)$ under the $U(1) \times U(1)$ gauge symmetry.
- $N = \tilde{N} = 0$ with $n = 0$ and $m = \pm 2$. This is a winding mode of the tachyon field. They are scalars in spacetime with charges $(0, \pm 2)$ under $U(1) \times U(1)$.

- $N = 1$ and $\tilde{N} = 0$ with $n = m = \pm 1$. These are two new spin 1 fields, $\alpha^\mu_{-1}\,|0;p\rangle$. They carry charge $(\pm 1, \pm 1)$ under the two $U(1) \times U(1)$.
- $N = 1$ and $\tilde{N} = 0$ with $n = -m = \pm 1$. These are a further two spin 1 fields, $\tilde{\alpha}^\mu_{-1}\,|0;p\rangle$, with charge $(\pm 1, \mp 1)$ under $U(1) \times U(1)$.

How do we interpret these new massless states? Let's firstly look at the spin 1 fields. These are charged under $U(1) \times U(1)$. As we mentioned in Section 7.7, the only way to make sense of charged massless spin 1 fields is in terms of a non-Abelian gauge symmetry. Looking at the charges, we see that at the critical radius $R = \sqrt{\alpha'}$, the theory develops an enhanced gauge symmetry

$$U(1) \times U(1) \rightarrow SU(2) \times SU(2)$$

The massless scalars from the $N = \tilde{N} = 0$ now join with the previous scalars to form adjoint representations of this new symmetry. We move away from the critical radius by changing the vacuum expectation value for σ. This breaks the gauge group back to the Cartan subalgebra by the Higgs mechanism.

From the discussion above, it's clear that this mechanism for generating non-Abelian gauge symmetries relies on the existence of the tachyon. For this reason, this mechanism doesn't work in Type II superstring theories. However, it turns out that it does work in the heterotic string, even though it has no tachyon in its spectrum.

8.3 Why Big Circles are the Same as Small Circles

The formula (8.6) has a rather remarkable property: it is invariant under the exchange

$$R \;\leftrightarrow\; \frac{\alpha'}{R} \tag{8.7}$$

if, at the same time, we swap the quantum numbers

$$m \;\leftrightarrow\; n \tag{8.8}$$

This means that a string moving on a circle of radius R has the same spectrum as a string moving on a circle of radius α'/R. It achieves this feat by exchanging what it means to wind with that it means to move.

As the radius of the circle becomes large, $R \rightarrow \infty$, the winding modes become very heavy with mass $\sim R/\alpha'$ and are irrelevant for the low-energy dynamics. But the momentum modes become very light, $M \sim 1/R$, and, in the strict limit form a continuum. From the perspective of the energy spectrum, this continuum of energy states is exactly what we mean by the existence of a non-compact direction in space.

In the other limit, $R \to 0$, the momentum modes become heavy and can be ignored: it takes way too much energy to get anything to move on the $\mathbf{S}^1$. In contrast, the winding modes become light and start to form a continuum. The resulting energy spectrum looks as if another dimension of space is opening up!

The equivalence of the string spectrum on circles of radii R and α'/R extends to the full conformal field theory and hence to string interactions. Strings are unable to tell the difference between circles that are very large and circles that are very small. This striking statement has a rubbish name: it is called *T-duality*.

This provides another mechanism in which string theory exhibits a minimum length scale: as you shrink a circle to smaller and smaller sizes, at $R = \sqrt{\alpha'}$, the theory acts as if the circle is growing again, with winding modes playing the role of momentum modes.

The New Direction in Spacetime

So how do we describe this strange new spatial direction that opens up as $R \to 0$? Under the exchange (8.7) and (8.8), we see that p_L and p_R transform as

$$p_L \to p_L \quad , \quad p_R \to -p_R$$

Motivated by this, we define a new scalar field,

$$Y^{25} = X_L^{25}(\sigma^+) - X_R^{25}(\sigma^-)$$

It is simple to check that in the CFT for a free, compact scalar field all OPEs of Y^{25} coincide with the OPEs of X^{25}. This is sufficient to ensure that all interactions defined in the CFT are the same.

We can write the new spatial direction Y directly in terms of the old field X, without first doing the split into left and right-moving pieces. From the definition of Y, one can check that $\partial_\tau X = \partial_\sigma Y$ and $\partial_\sigma X = \partial_\tau Y$. We can write this in a unified way as

$$\partial_\alpha X = \epsilon_{\alpha\beta}\, \partial^\beta Y \tag{8.9}$$

where $\epsilon_{\alpha\beta}$ is the antisymmetric matrix with $\epsilon_{\tau\sigma} = -\epsilon_{\sigma\tau} = +1$. (The minus sign from $\epsilon_{\sigma\tau}$ in the above equation is canceled by another from the Minkowski worldsheet metric when we lower the index on ∂^β).

The Shift of the Dilaton

The dilaton, or string coupling, also transforms under T-duality. Here we won't derive this in detail, but just give a plausible explanation for why it's the case. The main idea is that a scientist living in a stringy world shouldn't be able to do any experiments that distinguish between a compact circle of radius R and one of radius α'/R. But the first place you would look is simply the low-energy effective action which, working in Einstein frame, contains terms like

$$\frac{2\pi R}{2l_s^{24}g_s^2}\int d^{25}X\sqrt{-\tilde{G}}\; e^{\sigma}\,\mathcal{R}+\ldots$$

A scientist cannot tell the difference between R and $\tilde{R}=\alpha'/R$ only if the value of the dilaton is also ambiguous so that the term in front of the action remains invariant: i.e. $R/g_s^2 = \tilde{R}/\tilde{g}_s^2$. This means that, under T-duality, the dilaton must shift so that the coupling constant becomes

$$g_s \to \tilde{g}_s = \frac{\sqrt{\alpha'}g_s}{R} \tag{8.10}$$

8.3.1 A Path Integral Derivation of T-Duality

There's a simple way to see T-duality of the quantum theory using the path integral. We'll consider just a single periodic scalar field $X \equiv X + 2\pi R$ on the worldsheet. It's useful to change normalization and write $X = R\varphi$, so that the field φ has periodicity 2π. The radius R of the circle now sits in front of the action,

$$S[\varphi] = \frac{R^2}{4\pi\alpha'}\int d^2\sigma\;\partial_\alpha\varphi\,\partial^\alpha\varphi \tag{8.11}$$

The Euclidean partition function for this theory is $Z=\int\mathcal{D}\varphi\, e^{-S[\varphi]}$. We will now play around with this partition function and show that we can rewrite it in terms of new variables that describe the T-dual circle.

The theory (8.11) has a simple shift symmetry $\varphi\to\varphi+\lambda$. The first step is to make this symmetry local by introducing a gauge field A_α on the worldsheet which transforms as $A_\alpha \to A_\alpha - \partial_\alpha\lambda$. We then replace the ordinary derivatives with covariant derivatives

$$\partial_\alpha\varphi \to \mathcal{D}_\alpha\varphi = \partial_\alpha\varphi + A_\alpha$$

This changes our theory. However, we can return to the original theory by adding a new field, θ which couples as

$$S[\varphi,\theta,A] = \frac{R^2}{4\pi\alpha'}\int d^2\sigma\;\mathcal{D}_\alpha\varphi\,\mathcal{D}^\alpha\varphi + \frac{i}{2\pi}\int d^2\sigma\;\theta\,\epsilon^{\alpha\beta}\partial_\alpha A_\beta \tag{8.12}$$

The new field θ acts as a Lagrange multiplier. Integrating out θ sets $\epsilon^{\alpha\beta}\partial_\alpha A_\beta = 0$. If the worldsheet is topologically $\mathbf{R}^2$, then this condition ensures that A_α is pure gauge which, in turn, means that we can pick a gauge such that $A_\alpha = 0$. The quantum theory described by (8.12) is then equivalent to that given by (8.11).

Of course, if the worldsheet is topologically $\mathbf{R}^2$ then we're missing the interesting physics associated to strings winding around φ. On a non-trivial worldsheet, the condition $\epsilon^{\alpha\beta}\partial_\alpha A_\beta = 0$ does not mean that A_α is pure gauge. Instead, the gauge field can have non-trivial holonomy around the cycles of the worldsheet. One can show that these holonomies are gauge trivial if θ has periodicity 2π. In this case, the partition function defined by (8.12),

$$Z = \frac{1}{\text{Vol}}\int \mathcal{D}\varphi\mathcal{D}\theta\mathcal{D}A\ e^{-S[\varphi,\theta,A]}$$

is equivalent to the partition function constructed from (8.11) for worldsheets of any topology.

At this stage, we make use of a clever and ubiquitous trick: we reverse the order of integration. We start by integrating out φ which we can do by simply fixing the gauge symmetry so that $\varphi = 0$. The path integral then becomes

$$Z = \int \mathcal{D}\theta\mathcal{D}A\ \exp\left(-\frac{R^2}{4\pi\alpha'}\int d^2\sigma\ A_\alpha A^\alpha + \frac{i}{2\pi}\int d^2\sigma\ \epsilon^{\alpha\beta}\,(\partial_\alpha\theta)A_\beta\right)$$

where we have also taken the opportunity to integrate the last term by parts. We can now complete the procedure and integrate out A_α. We get

$$Z = \int \mathcal{D}\theta\ \exp\left(-\frac{\tilde{R}^2}{4\pi\alpha'}\int d^2\sigma\ \partial_\alpha\theta\,\partial^\alpha\theta\right)$$

with $\tilde{R} = \alpha'/R$ the radius of the T-dual circle. In the final integration, we threw away the overall factor in the path integral, which is proportional to $\sqrt{\alpha'}/R$. A more careful treatment shows that this gives rise to the appropriate shift in the dilaton (8.10).

8.3.2 T-Duality for Open Strings

What happens to open strings and D-branes under T-duality? Suppose firstly that we compactify a circle in direction X transverse to the brane. This means that X has Dirichlet boundary conditions

$$X = \text{const} \quad\Rightarrow\quad \partial_\tau X^{25} = 0 \qquad \text{at } \sigma = 0, \pi$$

But what happens in the T-dual direction Y? From the definition (8.9) we learn that the new direction has Neumann boundary conditions,

$$\partial_\sigma Y = 0 \quad \text{at } \sigma = 0, \pi$$

We see that T-duality exchanges Neumann and Dirichlet boundary conditions. If we dualize a circle transverse to a Dp-brane, then it turns into a D$(p+1)$-brane.

The same argument also works in reverse. We can start with a Dp-brane wrapped around the circle direction X, so that the string has Neumann boundary conditions. After T-duality, (8.9) changes these to Dirichlet boundary conditions and the Dp-brane turns into a D$(p-1)$-brane, localized at some point on the circle Y.

In fact, this was how D-branes were originally discovered: by following the fate of open strings under T-duality.

8.3.3 T-Duality for Superstrings

To finish, let's nod one final time towards the superstring. It turns out that the ten-dimensional superstring theories are not invariant under T-duality. Instead, they map into each other. More precisely, Type IIA and IIB transform into each other under T-duality. This means that Type IIA string theory on a circle of radius R is equivalent to Type IIB string theory on a circle of radius α'/R. This dovetails with the transformation of D-branes, since type IIA has Dp-branes with p even, while IIB has p odd. Similarly, the two heterotic strings transform into each other under T-duality.

8.3.4 Mirror Symmetry

The essence of T-duality is that strings get confused. Their extended nature means that they're unable to tell the difference between big circles and small circles. We can ask whether this confusion extends to more complicated manifolds. The answer is yes. The fact that strings can see different manifolds as the same is known as *mirror symmetry*.

Mirror symmetry is cleanest to state in the context of the Type II superstring, although similar behaviour also holds for the heterotic strings. The simplest example is when the worldsheet of the string is governed by a superconformal non-linear sigma-model with target space given by some Calabi-Yau manifold $\mathbf{X}$. The claim of mirror symmetry is that this CFT is identical to the CFT describing the string moving on a different Calabi-Yau manifold $\mathbf{Y}$. The topology of $\mathbf{X}$ and $\mathbf{Y}$ is not the same. Their Hodge diamonds are the mirror of each other; hence the name. The subject of mirror symmetry is an active area of research in geometry and provides a good example of the impact of string theory on mathematics.

8.4 Epilogue

We are now at the end of this introductory course on string theory. We began by trying to make sense of the quantum theory of a relativistic string moving in flat space. It is, admittedly, an odd place to start. But from then on we had no choices to make. The relativistic string leads us ineluctably to conformal field theory, to higher dimensions of spacetime, to Einstein's theory of gravity at low-energies, to good UV behaviour at high-energies and to Yang-Mills theories living on branes. There are few stories in theoretical physics where such meagre input gives rise to such a rich structure.

This journey continues. There is one further ingredient that it is necessary to add: supersymmetry. Even this is in some sense not a choice, but is necessary to remove the troublesome tachyon that plagued these lectures. From there we may again blindly follow where the string leads, through anomalies (and the lack thereof) in ten dimensions, to dualities and M-theory in eleven dimensions, to mirror symmetry and moduli stabilization and black hole entropy counting and holography and the miraculous AdS/CFT correspondence.

However, the journey is far from complete. There is much about string theory that remains to be understood. This is true both of the mathematical structure of the theory and of its relationship to the world that we observe. The problems that we alluded to in Section 6.4.5 are real. Non-perturbative completions of string theory are only known in spacetimes which are asymptotically anti-de Sitter, but cosmological observations suggest that our home is not among these. In attempts to make contact with the standard models of particle physics and cosmology, we typically return to the old idea of Kaluza-Klein compactifications. Is this the right approach? Or are we missing some important and subtle conceptual ingredient? Or is the existence of this remarkable mathematical structure called string theory merely a red-herring that has nothing to do with the real world?

In the years immediately after its birth, no one knew that string theory was a theory of strings. It seems very possible that we're currently in a similar situation. When the theory is better understood, it may have little to do with strings. We are certainly still some way from answering the simple question: what is string theory really?