The Physics of Floating

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1 Introduction

It seems to me highly inappropriate that a subject which excited Archimedes to the point where he supposedly felt compelled to cry "Eureka!" and exhibit himself through the streets of Syracuse appears never to have been the subject of an article in *Eureka*, the journal of the Archimedeans. Perhaps this has been justified on the grounds that Archimedes' Principle is so simple that we were all convinced in primary school that our teacher had taught us everything there could possibly be to know about it. On the contrary, there remains much to discover if only we are prepared to focus at the scale of bubbles floating in wine and breakfast cereal floating in milk rather than an ancient Greek getting into his bath. This article is aimed at redressing this imbalance by reviewing what we might all have learnt a long time ago were we a much smaller pond-surface-dwelling species. At such a scale, a new force would enter our lives: surface tension. Of course surface tension is not really new to us, but some of its effects certainly would be: objects much denser than water are able to remain floating (provided that they are small enough) and small objects that are less than a few millimetres apart are subject to an exotic force that cause them to clump together with similar objects.

For most pond-walking creatures, surface tension is what prevents them from drowning, since it provides a restoring force sufficient to overcome the animals' weights, allowing them to remain at the interface between air and water. The vertical deformation $z = h(x, y)$ of an interface from the flat due to the presence of such an object is determined by the requirement that the pressure drop across a deformed interface (proportional to the curvature of the deformation) must counteract the hydrostatic pressure brought about by that displacement. Mathematically, this is expressed by the Laplace-Young equation

$$\gamma \nabla \cdot \mathbf{n} = -\rho g h, \qquad (1)$$

where $\mathbf{n}(x, y)$ is the unit normal to the interface, γ the surface tension coefficient of the liquid-gas interface, ρ the density of the liquid and g the acceleration due to gravity. Using $\mathbf{n} = \nabla(z - h)/|\nabla(z - h)|$ with all lengths non-dimensionalised by the *capillary length*, $L_c = \sqrt{\gamma/\rho g}$, and writing $H = h/L_c$, equation (1) takes the form

$$\frac{\nabla^2 H}{(1 + H'^2)^{3/2}} = H \qquad (2)$$

where $H = h/L_c$ and (2) is to be solved with the boundary conditions that $H(\pm\infty) = 0$ and that the angle that the meniscus makes to the solid boundary is some fixed constant, θ, known as the contact angle.

2 A generalisation of Archimedes' Principle

If we imagine ourselves in the miniature realm of the pond skater, we realise that a floating object displaces liquid not only because of the excluded volume effect (with which we are familiar from school) but also because of the interfacial deformation that it causes elsewhere. What would Archimedes have made of this situation?

Keller [2] and Mansfield *et al.* [3] have considered this question only surprisingly recently and have shown that the vertical force on an interfacial object is equal to the weight of the total volume of liquid displaced by the object, including that displaced in the meniscus away from the body itself. Here we shall content ourselves with dividing the total force on the floating object into two components: $\mathbf{F}_{st}$, coming from the surface tension force acting at the contact line where the gas, liquid and solid meet, and $\mathbf{F}_{hp}$, which results from the action of the (hydrostatic) pressure on the parts of the object's surface that are wet. The first of these gives a force per unit length of the contact line equal to γ in the direction tangent to the interface itself, while the second is given by

$$\mathbf{F}_{hp} = \int_S \rho g z \mathbf{N}\,\mathrm{d}A, \tag{3}$$

where S is the surface wetted by the liquid, $\mathbf{N}$ is the normal pointing from that surface into the liquid and $\mathrm{d}A$ is an area element on the surface. The vertical component of this force is given by

$$\hat{\mathbf{e}}_z \cdot \mathbf{F}_{hp} = \int_S \rho g z \hat{\mathbf{e}}_z \cdot \mathbf{N}\,\mathrm{d}A = \int_A \rho g z\,\mathrm{d}x\,\mathrm{d}y, \tag{4}$$

where we have used the result that $\hat{\mathbf{e}}_z \cdot \mathbf{N}\,\mathrm{d}A$ is the projection of the surface S onto the x-y plane, denoted by A. Physically, the r.h.s. of (4) corresponds to the weight of liquid displaced between the wetted region of the object and the x-y plane tangent to the undeformed interface.

3 Floating versus sinking

As a simple application of Archimedes' Principle with the additional complication of surface tension, we next consider a question of considerable importance to creatures that live on the surfaces of ponds everywhere, namely "how heavy can an object of a given size be before it will sink?". Clearly, the detailed morphology of the object will be an important factor in reality, but here we consider a toy problem that is easily tractable at the expense of being highly idealised. We consider a single cylinder of infinite length lying horizontally at a liquid-gas interface as represented in Figure 1. The cylinder has a density ρ_s, which enters the calculation only through $D = \rho_s/\rho$, its value relative to the liquid density ρ. The other parameters of interest are the contact angle θ, which is a chemical property of the three phases that meet at the contact line, and the radius of the cylinder, R. The radius R is non-dimensionalised by the capillary length L_c and is represented by the *Bond number* $B = R^2/L_c^2$ for historical reasons. For small Bond numbers the effects of surface tension are important, but for large Bond numbers (a radius of a few centimetres or more for an air-water interface) the radius is so large that surface tension effects are negligible.

We shall determine the maximum density ratio, $D_{\max}$, that a cylinder with a given Bond number and contact angle can have before it sinks, but before embarking on the detailed calculation we investigate the limits $B \ll 1$ and $B \gg 1$ physically. In the latter case, we expect the influence of surface tension to be negligible and so only objects with density less than or equal to the density of the liquid can float, i.e., for $B \gg 1$

$$D_{\max} \sim 1. \tag{5}$$

For small objects, we expect the Archimedean buoyancy to be negligible and the object to float purely by virtue of the vertical component of surface tension at the contact line. Per unit length, the maximum force that can be generated via this mechanism is 2γ, which must

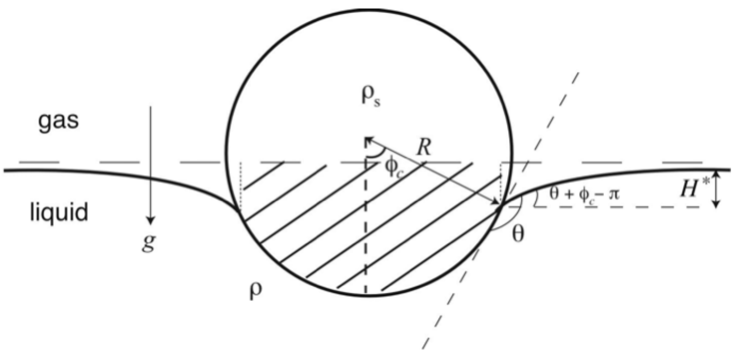

Figure 1: Setup and notation for a circular cylinder floating horizontally at the interface between a liquid and gas. The hatched area represents the displaced liquid whose weight is equivalent to the buoyancy force due to hydrostatic pressure on the cylinder.

balance the weight of the cylinder, namely $\rho_s g\pi R^2$. This simple argument gives that for $B \ll 1$

$$D_{\max} \sim \frac{2}{\pi B}. \tag{6}$$

Having determined the asymptotic behaviour of $D_{\max}$ for extreme values of B it remains to calculate its behaviour for intermediate B. Mansfield *et al.* [3] have shown that it is possible to provide a lower bound on $D_{\max}$ by approximating the interfacial gradients as being small. For a full treatment of this problem, however, it is not enough to make such an assumption, since just before sinking we should expect the interface to be subject to large deformations beyond the regime of this linear theory. In the slightly contrived geometry chosen here, we are able to make progress without this assumption since the Laplace-Young equation (2) for the interfacial deformation takes the relatively simple form

$$H = \frac{H''}{(1+H'^2)^{3/2}}, \tag{7}$$

which can be integrated once subject to the requirement that $H(\pm\infty) = H'(\pm\infty) = 0$ to give

$$H^2 = 2\left(1 - (1+H'^2)^{-1/2}\right). \tag{8}$$

Using (8) and the other boundary condition that the interface makes an angle $\theta + \phi_c - \pi$ with the horizontal at the contact line, the interfacial deformation at the contact line, H^*, is

$$|H^*| = \sqrt{2(1 - |\cos(\theta+\phi_c)|)}, \tag{9}$$

where ϕ_c is as defined in Figure 1 and we have had to be careful in choosing the correct branch of the cosine. With this result, the vertical force balance condition becomes (in non-dimensional terms)

$$f(\phi_c) \equiv 2\sin\phi_c\sqrt{2B(1-|\cos(\theta+\phi_c)|)} + B(\phi_c - \sin\phi_c\cos\phi_c) - 2\sin(\theta+\phi_c) = \pi B D. \tag{10}$$

Here, the r.h.s. of (10) is the non-dimensional weight (per unit length) of the cylinder, which must be balanced by the weight of displaced liquid in the hatched area of Figure 1 (the first two terms on the l.h.s.) and the vertical component of the surface tension force (the third term on the l.h.s.). To determine the maximum density that a cylinder may have without sinking, we must find the maximum value of $f(\phi_c)$ for a given value of θ and B. Unfortunately,

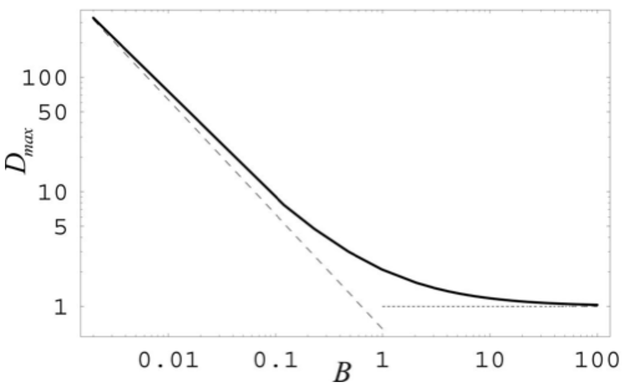

Figure 2: Solid line shows the numerically computed maximum density, $D_{\max}$, of a floating cylinder with contact angle $\theta = 2\pi/3$ as a function of the Bond number, B. The short and long dashed lines represent the asymptotic results (5) and (6) respectively.

this can only be done numerically, with the results for $\theta = 2\pi/3$ and different cylinder Bond numbers plotted in Figure 2 along with the asymptotic results (5) and (6). The asymptotic results are indeed borne out by the numerical calculation for extreme values of B, but for $B \in (0.1, 10)$ we see that the combined effects of surface tension and hydrostatic pressure are enough to support considerably denser objects than either could alone.

There appears to be little experimental data to compare these results with – particularly for intermediate Bond numbers. However, Gao and Jiang [1] recently measured the vertical force that can be supplied by a water strider's leg before it sinks. In these experiments, $B \approx 0.01$ and the leg length is around 9mm so that the theory above predicts a vertical force of around 143 dynes, which compares reasonably favourably with the 152 dynes measured experimentally.

4 How (not) to walk on water

Having seen that for objects of very small Bond number the weight that can be supported is proportional to the wetted perimeter of the floating object, it is natural to consider whether this can be used to circumvent the normal physics that prevents humans from walking on water. In particular, it seems possible that by using special shoes with a fractal outline, the Koch snowflake, for example, one might be able to make shoes that would allow us to perform such a feat. Before you rush off to try this, however, bear in mind the generalisation of Archimedes' Principle discussed earlier: the maximum upward force that can be produced is equal to the *total* weight of liquid that is displaced (including that within the interfacial menisci). Since the density of humans is roughly that of water, fractal shoes with a cross-sectional area typical of ordinary shoes would thus have to displace a volume of water roughly equal to the volume of the human they are supposed to support. Since interfacial deformations extend away from the object only a few times the capillary length and can only extend to depths of the same order at the edge of the floating object, that volume would predominantly come from the depth to which the shoes would fall (without sinking!), requiring a very odd shoe design and the wearer only just to have their head above the normal surface of the water, if at all. If you wish to try your luck making such a pair of shoes, then you have been warned – you may well be able to stay dry…just.

References

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[3] E.H. Mansfield, H.R. Sepangi and E.A. Eastwood, "Equilibrium and mutual attraction or repulsion of objects supported by surface tension", *Phil. Trans. R. Soc. Lond.* A **355** (1997) 869–919