

# More Properties of Pythagorean Triples

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## Introduction

Pythagoras' Theorem is a result known by almost every secondary school child around the world. Well known also is the existence of so-called 'Pythagorean Triples', integer or 'Diophantine' solutions to Pythagoras' famous equation.

There are many ways of generating these triples. Perhaps the best known is Newton's method which relies on the identity:

$$(x^2 - y^2)^2 + (2xy)^2 \equiv (x^2 + y^2)^2$$

This identity clearly suggests that the integer solutions to the equation  $a^2 + b^2 = c^2$  are of the form:

$$a = d(x^2 - y^2), b = 2dxy, c = d(x^2 + y^2), \text{ with } x > y > 0.$$

Where  $(x, y) = 1$ ,  $x$  and  $y$  are of opposite parity and  $(a, b, c) = d$ . It can be proved<sup>1</sup> that every Pythagorean Triple can be written in this way and so it will be useful to look at these  $a$ ,  $b$  and  $c$ . If  $d = 1$ , then we say that the Triple is Primitive. We begin with a result we suspect is well known but has a more interesting corollary.

### Theorem 1

*If  $P$  is the perimeter of a Pythagorean Triangle with integral sides  $a$ ,  $b$  and  $c$  then:  $P \mid ab$ .*

*Proof:* Using Newton's method to express the sides  $a$ ,  $b$  and  $c$  in terms of the parameters  $x$  and  $y$ , we have:

$$P = a + b + c = d(x^2 - y^2 + 2xy + x^2 + y^2) = 2dx(x + y).$$

$$ab = 2d^2xy(x^2 - y^2) = 2d^2x(x + y)y(x - y) = Pdy(x - y).$$

This proves the theorem. ♣

A useful corollary of this theorem is the following:

### Corollary

*If a Pythagorean Triangle has even sides, then its perimeter,  $P$ , divides its area,  $A$ .*

*Proof:* We have  $d = 2k$  say, as we know that all the sides are even. As  $A = \frac{1}{2} ab$ , we have (from the proof of Theorem 1) that:

$A = Pky(x - y)$  i.e.  $P \mid A$ . The corollary follows. ♣

It is interesting to have an algorithm for deciding, given a primitive triple  $u, v, w$ , which of  $a, b$  and  $c$  these correspond to. Clearly,  $c = \max \{u, v, w\}$ . We then change labels so that our triple becomes  $\{r, s, c\}$  where the smaller two of the original triple have simply been relabelled. If we now go back to the original notation of Newton's method introduced earlier then we have:

$$c - a = 2y^2, c - b = (x - y)^2.$$

Clearly  $c - a$  is not a perfect square (since  $\sqrt{2}$  is irrational) while  $c - b$  is. This then gives us our algorithm as we conclude that:

$$b = \begin{cases} r & \text{if } c - r \text{ is a perfect square} \\ s & \text{if } c - r \text{ is \textbf{not} a perfect square} \end{cases}$$

The remaining side being put equal to  $a$ . If the triple is not primitive then we divide the triple by  $d$  and then apply this algorithm to the resulting primitive triple. Multiplying up by  $d$  then gives us  $a, b$  and  $c$ . We can strengthen the Corollary to Theorem 1 by using this algorithm and obtain the following Theorem:

## Theorem 2

*The perimeter of a Pythagorean Triangle divides its area iff  $4 \mid (c - a)$  or  $4 \mid (c - b)$ .*

*Proof:* There are two cases to consider:

i)  $4 \mid c - a$ .

Here:  $c - a = 2dy^2$  so that either  $2 \mid y$  or  $2 \mid d$ . If  $2 \mid d$  then the all the sides of the triangle are even and the Corollary to Theorem 1 applies. If  $2 \mid y$  then we have  $y = 2k$  say, and so  $A = xy(x - y)(x + y) = 2x(x + y)k(x - y) = Pk(x - y)$ .

ii)  $4 \mid c - b$ .

Here:  $c - b = d(x - y)^2$  so that either  $2 \mid x - y$  or  $4 \mid d$ . If  $4 \mid d$  the sides are all even and so once again the Corollary to Theorem 1 applies. If  $2 \mid x - y$  then we can write  $x - y = 2k$  giving us  $A = Pky$ . In both (i) and (ii) we have  $P \mid A$ .

Conversely, if  $P \mid A$  then we have:

$$\frac{A}{P} = \frac{dy(x + y)}{2} = k, \text{ say. Thus } d \text{ is even or } y \text{ is even or } x \text{ and } y \text{ are both odd.}$$

If  $d$  or  $y$  is even then  $4 \mid 2dy^2 = c - a$  and if  $x$  and  $y$  are both odd then  $4 \mid d(x - y)^2 = c - b$ . ♣

## Remarks

It is important that we have this Theorem as the Corollary to Theorem 1 clearly doesn't pick up any Primitive Pythagorean Triples where the perimeter divides the area such as

{5, 12, 13}. The corollary to Theorem 1 is also a special case of Theorem 2 as we have seen in the proof of Theorem 2.

If we look at a table of  $a$ ,  $b$  and  $c$  for a few values of  $x$  and  $y$ , then other patterns emerge:

$x$	$y$	$a (= x^2 - y^2)$	$b (= 2xy)$	$c (= x^2 + y^2)$
2	1	3	4	5
3	2	5	12	13
4	3	7	24	25
5	4	9	40	41
6	5	11	60	61
7	6	13	84	85
8	7	15	112	113
9	8	17	144	145
10	9	19	180	181

**Table 1: Examples of Pythagorean Triples**

It seems that the hypotenuse of each triple is the shortest side of another triple whose longest sides differ by 1. We can do better than this, however, as it also seems likely that almost every number is the shortest side of a Pythagorean Triple. This can be expressed as a Theorem:

**Theorem 3**

*If  $n = 1, 2$  or  $4$  there is no Pythagorean Triangle with  $n$  as its shortest side. Otherwise if  $n$  odd or if  $4 \mid n$  there is a Primitive Pythagorean Triangle with  $n$  as its shortest side otherwise there is an Non-primitive Pythagorean Triangle (with  $d = 2$ ) with  $n$  as its shortest side.*

*Proof:* If  $n = 1$ , we would need  $c^2 = 1 + b^2$ . This is impossible since no two squares differ by 1 (except 0 and 1 which gives us a shortest side of 0!). With  $n = 2$ , we would need  $c^2 = 4 + b^2$  i.e. either  $c = 2, b = 0$  or  $c - b = 1, c + b = 4$  giving  $c = 2\frac{1}{2}, b = 1\frac{1}{2}$ . The case  $n = 4$  is a little harder still but we know that it is to be a Primitive Triangle (if it were not we would have a Triple with 1 or 2 as shortest side). Thus we need  $4 = 2xy$  so that  $x = 2, y = 1$  are the only possibility. Clearly, however, this gives the triple {3, 4, 5} and so there is no Triangle with 4 as shortest side.

We look at the case where  $n$  odd and  $n > 1$  and let:

$$y = \frac{n-1}{2}, x = \frac{n+1}{2}$$

$$\Rightarrow 2xy = \frac{(n-1)(n+1)}{2} = \frac{n^2 - 1}{2}$$

$$x^2 + y^2 = \frac{n^2 + 1}{2} = 2xy + 1$$

$$x^2 - y^2 = n$$

This triple of numbers is a Pythagorean triple that meets the criteria given (it is primitive since two of the sides differ by 1 and so cannot have a common factor). The odd number is 'forced' to be the smallest side by the fact that the other two sides differ by 1.

We now look at the case  $n$  even in two stages. If  $4 \mid n$ , then we look at the triple of numbers  $\{2x, x^2 - 1, x^2 + 1\}$  where  $n = 2x$ . Now, as  $x$  is even, we know that  $x^2 - 1$  and  $x^2 + 1$  are odd and thus are coprime. Thus the triple we have is Primitive and is clearly a Pythagorean Triple, it remains to show that  $2x$  is the smaller of  $2x$  and  $x^2 - 1$ . But we know that for  $x > 2$ ,  $(x - 1)^2 > 2$  and so  $x^2 - 1 > 2x$  and we conclude that if  $4 \mid n$  then there is a Primitive Pythagorean Triangle with  $n$  as its shortest side.

The only remaining possibility is that  $n = 2(2k + 1)$ . By the first part of the Theorem, we know that there is a Pythagorean Triangle with  $2k + 1$  as its shortest side and thus (upon multiplying the triple by 2 giving  $d = 2$ ) there is a Pythagorean Triangle with  $n$  as its shortest side. We must also show that this is the 'best' that we can do i.e. that there is no Primitive Pythagorean Triangle with  $n$  as its shortest side. With a Primitive Triangle, we would have  $d = 1$ , and so we need either  $x^2 - y^2 = 2(2k + 1)$  or  $2xy = 2(2k + 1)$ . The first case cannot occur since  $x$  and  $y$  are of opposite parity and so  $x^2 - y^2$  is inevitably odd. Similarly, the second case cannot occur since we would need  $xy = 2k + 1$  and we already know that one of  $x$  and  $y$  is even. ♣

## Reference

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<sup>1</sup> Hardy, G H and Wright, E M. (1960). An Introduction to the Theory of Numbers. Oxford University Press.

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