The trailing vorticity field behind a point source in 2D incompressible linear shear flow†

SJOERD W. RIENSTRA[1] ‡, **MIRELA DARAU**[1,2] AND **EDWARD J. BRAMBLEY**[3]

[1]Dept of Maths and Comp Science, TU Eindhoven, The Netherlands

[2]Dept of Maths and Comp Science, WU of Timişoara, Romania

[3]DAMTP, University of Cambridge, UK

(Received 17 December 2011)

The explicit exact analytic solution for harmonic perturbations from a point mass source in an incompressible inviscid two-dimensional linear shear is derived by using a Fourier transform method. The two cases of an infinite shear flow and a semi-infinite shear flow over an impedance boundary are both considered. Since the pressure field is logarithmically diverging, its Fourier representation involves a diverging integral that is interpreted as an integral of generalised functions. The dominant feature of the solution is the hydrodynamic wake caused by the shed vorticity of the source. For linear shear over an impedance boundary, in addition to the wake (at most) two surface modes along the wall are possible. The implications for duct acoustics with flow over an impedance wall are discussed.

Key words: Boundary layer, impedance wall

1. Introduction

The sound field from a mass point source in a cylindrical duct with a uniform mean flow with finite linearly-varying boundary layers (as studied in Brambley *et al.* (2011b)) has been shown to have a non-modal contribution, triggered by a source within the shear layer. This field is not present when the mass source is within the uniform flow region. The results derived in this paper inform the identification of this non-modal contribution as a wake, non-acoustic and hydrodynamic in nature, due to the shed vorticity of the mass source.

In the following, we consider the effect of a time-harmonic point mass source on an incompressible inviscid two-dimensional linear shear of infinite (section 3) or semi-infinite (section 4) extent. The explicit exact analytic solutions obtained appear to be new, in spite of this rather simple configuration, with the nearest known solution being the velocity field given in Criminal & Drazin (1990) for the initial value problem of an impulsive point source in a linear shear layer. The solutions are obtained using a classical Fourier transform method, leading to integrals which are not normally convergent and which must therefore be interpreted as integrals of generalised functions Jones (1982).

‡ Corresponding author. Email: S.W.Rienstra@tue.nl

† A preliminary version of this work Brambley *et al.* (2011a) was presented as part of paper AIAA-2011-2806 at the 17th AIAA/CEAS Aeroacoustics Conference.

For the semi-infinite case, we take an infinite shear layer over an impedance wall such that the mean flow vanishes at the wall. This problem is in many respects similar to the infinite shear case, since there is again the vorticity trailing from the source, but at the same time the interaction with the wall is more subtle.

2. A time-harmonic line source in a shear flow: general remarks

Consider the equations for conservation of mass and momentum in a two dimensional inviscid flow with a mass source Q,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{v}) = Q, \qquad \rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p = 0. \tag{2.1}$$

When the source varies harmonically in time with a small amplitude, and the mean flow is a parallel flow with velocity $\boldsymbol{v} = U(y)\boldsymbol{e}_x$ and uniform pressure p_0, density ρ_0 and soundspeed c_0, we linearize the problem by introducing perturbations in the usual complex form

$$\rho = \rho_0 + c_0^{-2} p\,\mathrm{e}^{\mathrm{i}\omega t},\ \boldsymbol{v} = (U(y) + u\,\mathrm{e}^{\mathrm{i}\omega t})\boldsymbol{e}_x + v\,\mathrm{e}^{\mathrm{i}\omega t}\,\boldsymbol{e}_y,\ p = p_0 + p\,\mathrm{e}^{\mathrm{i}\omega t},\ Q = q\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{2.2}$$

to obtain

$$\begin{aligned} \frac{1}{c_0^2}\left(\mathrm{i}\omega + U\frac{\partial}{\partial x}\right)p + \rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= q \\ \rho_0\left(\mathrm{i}\omega + U\frac{\partial}{\partial x}\right)u + \rho_0\frac{\mathrm{d}U}{\mathrm{d}y}v + \frac{\partial p}{\partial x} &= 0 \\ \rho_0\left(\mathrm{i}\omega + U\frac{\partial}{\partial x}\right)v + \frac{\partial p}{\partial y} &= 0 \end{aligned} \tag{2.3}$$

When the mean flow is of uniform vorticity $-\sigma$ and given by

$$U(y) = U_0 + \sigma y, \tag{2.4}$$

we obtain, after taking the curl of the momentum equation,

$$\rho_0\left(\mathrm{i}\omega + U\frac{\partial}{\partial x}\right)\left(\chi + \frac{\sigma}{\rho_0 c_0^2}p\right) = \sigma q. \tag{2.5}$$

where $\chi = \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u$ is the z-component of vorticity perturbation. We see that the mass source may act as a source of vorticity by a coupling with the mean flow vorticity. With a monopole type point source

$$q = \rho_0 S\delta(x)\delta(y),$$

we have

$$\left(\mathrm{i}\omega + U\frac{\partial}{\partial x}\right)\left(\chi + \frac{\sigma}{\rho_0 c_0^2}p\right) = \sigma S\delta(x)\delta(y) \tag{2.6}$$

which has, under causal free field conditions and $U_0 \neq 0$, the solution

$$\chi + \frac{\sigma}{\rho_0 c_0^2}p = \frac{\sigma S}{U_0}H(x)\,\mathrm{e}^{-\mathrm{i}k_0 x}\,\delta(y), \quad k_0 = \frac{\omega}{U_0}. \tag{2.7}$$

Noting that the pressure term in (2.7) cannot be discontinuous, we see that this simple derivation shows that a line source in linear shear flow produces a semi-infinite sheet of vorticity, undulating with hydrodynamic wave number k_0. This was observed in the acoustic problem of a source in the boundary layer of a mean flow in a duct Brambley *et al.* (2011b). Here we will show that this phenomenon is not essentially acoustical but more generally of hydrodynamic nature.

We will consider the above problem in two settings: under free field conditions, and in the neighbourhood of an impedance wall. We will derive analytically explicit solutions in pressure and velocity by assuming the fluid to be incompressible. In this case the factors proportional to $1/c_0^2$ reduce to zero and equation (2.7) gives the vorticity as

$$\chi = \frac{\sigma S}{U_0} H(x)\,\mathrm{e}^{-\mathrm{i}k_0 x}\,\delta(y). \tag{2.8}$$

We will solve the problem by Fourier transformation in x, as this approach is most flexible and versatile. It can be utilised to both the free field and the impedance wall configuration, and for the velocities and the pressure. In the free field problem, the velocities can be obtained also by a more direct integration of a Greens function representation, but this approach does not seem to be as convenient as Fourier transformation.

There is a catch however: the pressure field of a 2D source diverges like $\log(x^2+y^2)$ for large x^2+y^2, and is therefore not Fourier transformable. We will circumvent this by considering the divergent Fourier integral of the pressure in the context of generalised functions and carefully subtract the singular part. The result is then only unique up to addition of an undetermined constant. This, however, is not a problem, because the pressure in an incompressible model in general is only defined up to a (time-dependent) constant.

We finally note that we deliberately do not make the problem dimensionless. The nature of the problem (with an infinite linear mean shear flow) is such that there is no really characteristic length scale or velocity scale in the problem. Much depends on the relative position of the observer. The price is that we have to be careful in drawing conclusions about trends and effects of parameters, and we always should consider dimensionless combinations.

3. A time-harmonic point mass source in infinite linear shear

3.1. *Free field solution*

Consider the two-dimensional incompressible inviscid model problem of perturbations of a linear sheared mean flow due to a time-harmonic point source at $x=y=0$ with time dependence $\mathrm{e}^{\mathrm{i}\omega t}$

$$\begin{aligned}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= S\delta(x)\delta(y),\\ \rho_0\left(\mathrm{i}\omega + U\frac{\partial}{\partial x}\right)u + \rho_0\frac{\mathrm{d}U}{\mathrm{d}y}v + \frac{\partial p}{\partial x} &= 0,\\ \rho_0\left(\mathrm{i}\omega + U\frac{\partial}{\partial x}\right)v + \frac{\partial p}{\partial y} &= 0.\end{aligned} \tag{3.1}$$

The far field boundary conditions will be of vanishing velocity, but (as we will see) not of vanishing pressure. Another point to be noted here, as it will be important later, is that the pressure appears only as a spatial gradient, and so will necessarily only be determined up to a constant.

After Fourier transformation in x we obtain the following set of equations

$$-\mathrm{i}k\tilde{u} + \tilde{v}' = S\delta(y), \qquad \mathrm{i}\rho_0\Omega\tilde{u} + \rho_0 U'\tilde{v} - \mathrm{i}k\tilde{p} = 0, \qquad \mathrm{i}\rho_0\Omega\tilde{v} + \tilde{p}' = 0, \tag{3.2}$$

where $\Omega = \omega - kU$. This system may be further reduced to an incompressible form of the Pridmore-Brown (1958) equation by eliminating $\tilde{v}$ and $\tilde{u}$, which, upon considering a doubly-infinite linear shear flow with $U(y) = U_0 + \sigma y$, becomes

$$\tilde{p}'' + \frac{2k\sigma}{\Omega}\tilde{p}' - k^2\tilde{p} = -\mathrm{i}\rho_0 S\Omega_0\delta(y). \tag{3.3}$$

The boundary conditions will be a decaying field at infinity, although that will be strictly possible only for the velocity; the pressure will at best be slowly diverging.

The homogeneous equation has two independent solutions (Rayleigh (1945), Drazin & Reid (2004)), $\mathrm{e}^{\pm ky}(\Omega \pm \sigma)$, or

$$\tilde{p}_1(y) = \mathrm{e}^{|k|y}(\Omega + \mathrm{sign}(\mathrm{Re}\, k)\sigma), \quad \tilde{p}_2(y) = \mathrm{e}^{-|k|y}(\Omega - \mathrm{sign}(\mathrm{Re}\, k)\sigma), \tag{3.4}$$

where

$$|k| = \mathrm{sign}(\mathrm{Re}\, k)k = \sqrt{k^2}, \tag{3.5}$$

where $\sqrt{\ }$ denotes the principal value square root, and $|k|$ has thus branch cuts along $(-\mathrm{i}\infty, 0)$ and $(0, \mathrm{i}\infty)$. Note that neither of these solutions has a log-like singularity or requires a branch cut in the complex-y plane. The Wronskian is

$$W(y; k) = \tilde{p}_2'(y)\tilde{p}_1(y) - \tilde{p}_1'(y)\tilde{p}_2(y) = -2|k|\Omega^2, \tag{3.6}$$

and the Fourier transformed solution is thus

$$\tilde{p}(y, k) = \frac{\frac{1}{2}\mathrm{i}\rho_0 S}{|k|\Omega_0}\,\mathrm{e}^{-|ky|}\big(\Omega\Omega_0 - \sigma^2|ky| - \sigma^2\big). \tag{3.7}$$

The physical field in the x, y-domain is hence obtained by inverse Fourier transformation

$$p(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{p}(y, k)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k = \frac{\mathrm{i}\rho_0 S}{4\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}kx-|ky|}}{|k|\Omega_0}\big(\Omega\Omega_0 - \sigma^2|ky| - \sigma^2\big)\,\mathrm{d}k, \tag{3.8}$$

which has singularities at $k = 0$ (if $\omega^2 \neq \sigma^2$) and at $k = k_0 = \omega/U_0$ from $\Omega_0 = -U_0(k - k_0) = 0$. Folding the contour around the branch cuts of $|k|$ (upwards if $x < 0$ and downwards if $x > 0$) to obtain the steepest descent contour, while noting that the contribution of the k_0 pole is the downstream trailing vorticity of the point source, we obtain

$$p(x, y) = \frac{\rho_0 S}{2\omega}\sigma^2 H(x)(1 + k_0|y|)\,\mathrm{e}^{-\mathrm{i}k_0x - k_0|y|} + \frac{\mathrm{i}\rho_0 S}{2\pi}\int_0^{\infty}\frac{\mathrm{e}^{-\lambda|x|}}{\lambda\Omega_0^{\pm}}\big[\big(\Omega^{\pm}\Omega_0^{\pm} - \sigma^2\big)\cos\lambda y - \sigma^2\lambda y\sin\lambda y\big]\,\mathrm{d}\lambda \tag{3.9}$$

where $\Omega^{\pm} = \omega \pm \mathrm{i}\lambda U$, $\pm = \mathrm{sign}(x)$, $H(x)$ is Heaviside's step function.

The singularity at $k = 0$ is, unlike the one at k_0, not a pole and has a different origin. Due to this singularity the Fourier representation of the pressure is too singular to be interpreted normally. This is caused by p being not Fourier transformable, not because p itself is singular. As mentioned before, if $\omega^2 \neq \sigma^2$, p diverges as $\sim \log(x^2 + y^2)$ for $x^2 + y^2 \to \infty$ and is hence not integrable. This is an artefact of the line source model in an incompressible medium. When we consider the incompressible problem as an inner problem of a larger compressible problem, this diverging behaviour disappears as it changes in the far field into an outward radiating, and hence decaying, acoustic wave of some kind.

The inverse Fourier integral, however, can be found if the singular integral is interpreted in the generalised sense, and the singular part is split off. Following Jones (1982), p. 105), we change the semi-infinite integral into a doubly infinite one by replacing $1/\lambda$ by the generalised function

$$\lambda^{-1}H(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}H(\lambda)\log|\lambda|.$$

After integration by parts we obtain the convergent integrals

$$p(x,y) = \frac{\rho_0 S}{2\omega}\sigma^2 H(x)(1+k_0|y|)\,\mathrm{e}^{-\mathrm{i}k_0x-k_0|y|} - \frac{\mathrm{i}\rho_0 S}{2\pi}\int_0^\infty \log\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\mathrm{e}^{-\lambda|x|}\,\Omega^\pm\cos\lambda y\right]\mathrm{d}\lambda$$
$$+\frac{\mathrm{i}\rho_0 S}{2\pi}\sigma^2\int_0^\infty \log\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\mathrm{e}^{-\lambda|x|}\,\frac{\cos\lambda y+\lambda y\sin\lambda y}{\Omega_0^\pm}\right]\mathrm{d}\lambda \quad (3.10)$$

Each one can be integrated as follows

$$\int_0^\infty \log\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\mathrm{e}^{-\lambda|x|}\,\Omega^\pm\cos\lambda y\right]\mathrm{d}\lambda = \omega\gamma+\tfrac{1}{2}\omega\log(x^2+y^2)-\mathrm{i}U\frac{x}{x^2+y^2}$$
$$\omega\int_0^\infty \log\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\mathrm{e}^{-\lambda|x|}\,\frac{\cos\lambda y+\lambda y\sin\lambda y}{\Omega_0^\pm}\right]\mathrm{d}\lambda = \gamma+\tfrac{1}{2}\log(x^2+y^2)$$
$$+\tfrac{1}{2}(1-k_0y)E(k_0,z)+\tfrac{1}{2}(1+k_0y)E(k_0,\bar{z})$$

where $z = x+\mathrm{i}y$, $\gamma = 0.5772156649\ldots$ is Euler's constant and $E(k_0,z) = \mathrm{e}^{-\mathrm{i}k_0z}\,E_1(-\mathrm{i}k_0z)$, with E_1 the exponential integral with (here) the standard branch cut along the negative real axis of its argument. This results for $E_1(-\mathrm{i}k_0z)$ in a branch cut along the line $x=0, y>0$ and for $E_1(-\mathrm{i}k_0\bar{z})$ in a branch cut along the line $x=0, y<0$ (see Appendix, equation 5.2). Altogether, we have

$$p(x,y) = -\frac{\rho_0 S}{2\pi}(U_0+\sigma y)\frac{x}{x^2+y^2}+\frac{\mathrm{i}\rho_0 S\sigma}{2\pi}(\alpha-\alpha^{-1})\Big[\gamma+\tfrac{1}{2}\log(x^2+y^2)\Big]$$
$$+\frac{\mathrm{i}\rho_0 Sk_0\alpha}{4\pi}\Big[(\alpha U_0-\sigma y)E(k_0,z)+(\alpha U_0+\sigma y)E(k_0,\bar{z})$$
$$-2\pi\mathrm{i}H(x)(\alpha U_0+\sigma|y|)\,\mathrm{e}^{-\mathrm{i}k_0x-k_0|y|}\Big] \quad (3.11)$$

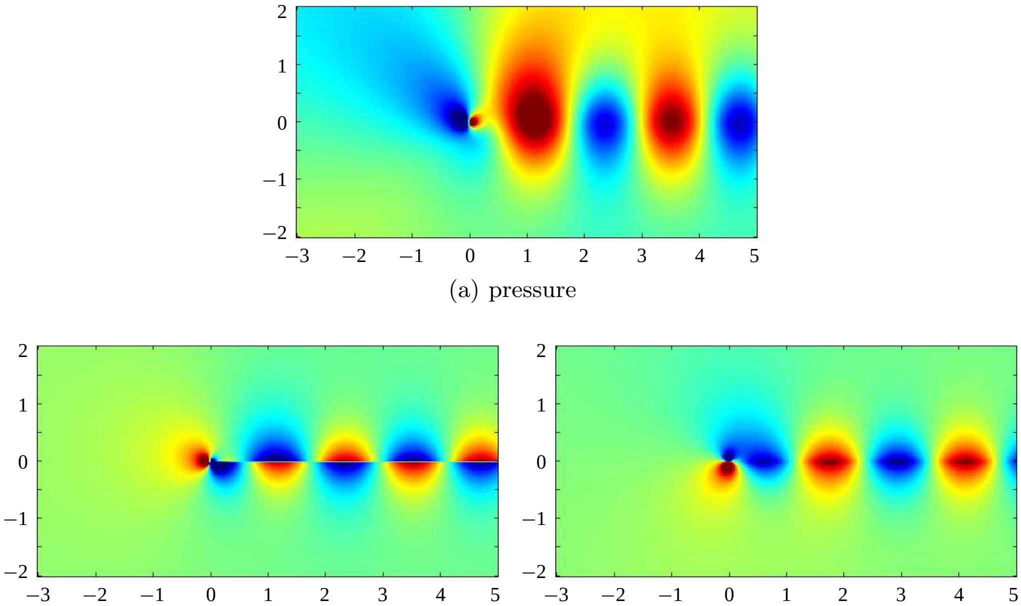
with

$$\alpha = \frac{\sigma}{\omega}. \quad (3.12)$$

A seemingly different result would have been obtained if we had scaled λ by a positive factor. The above regularisation of the divergent integral would have produced, via the logarithm, a result that differs by a constant. This, however, is not a problem because the pressure is only defined up to a constant in the first place. Indeed, the term in (3.11) proportional to γ is also not relevant and can be discarded. At the same time, this explains the at first sight dubious dimensional argument of the $\log(x^2+y^2)$ function.

As opposed to p, the integrals for v or u are convergent (outside the source) and can be found without resorting to generalised functions. We have

$$\tilde{v}(y,k) = \tfrac{1}{2}S\,\mathrm{e}^{-|ky|}\left(\mathrm{sign}(y)+\mathrm{sign}(\mathrm{Re}\,k)\frac{\sigma}{\Omega_0}\right),$$
$$\tilde{u}(y,k) = \tfrac{1}{2}\mathrm{i}S\,\mathrm{e}^{-|ky|}\left(\mathrm{sign}(\mathrm{Re}\,k)+\mathrm{sign}(y)\frac{\sigma}{\Omega_0}\right) \quad (3.13)$$



(a) pressure

(b) u-velocity

(c) v-velocity

FIGURE 1. Snapshots in time, free field. $\omega = 8$, $\sigma = 6$, $U_0 = 3$, giving $k_0 = 2.67$, $\alpha = 0.75$

and obtain

$$
\begin{aligned}
v(x, y) &= \frac{S}{2\pi} \frac{y}{x^2 + y^2} - \frac{Sk_0\alpha}{4\pi} \left[E(k_0, z) + E(k_0, \bar{z}) - 2\pi \mathrm{i} H(x)\, \mathrm{e}^{-\mathrm{i}k_0 x - k_0|y|} \right] \\
u(x, y) &= \frac{S}{2\pi} \frac{x}{x^2 + y^2} + \frac{\mathrm{i}Sk_0\alpha}{4\pi} \left[E(k_0, z) - E(k_0, \bar{z}) + 2\pi \mathrm{i} H(x) \operatorname{sign}(y)\, \mathrm{e}^{-\mathrm{i}k_0 x - k_0|y|} \right]
\end{aligned}
\tag{3.14}
$$

The branch cuts of the exponential integrals (in the E-functions) cancel the jumps due to the $H(x)$-terms, to produce continuous p and v fields. Only u has a tangential discontinuity along $y = 0, x > 0$, but this is due to the $\operatorname{sign}(y)$ term. This corresponds with the $\delta(y)$-function behaviour of the vorticity given in (2.8).

3.2. *An example*

A graphical example of this solution, in the form of snap shots in time of the pressure and velocity fields (i.e. including the factor $\mathrm{e}^{\mathrm{i}\omega t}$), is given in figure 1. The parameters chosen are $\omega = 8$, $\sigma = 6$, $U_0 = 3$, and hence $k_0 = 2.667$, $\alpha = 0.75$. In order to remove the effect of the undetermined constant, the plot-domain averaged value of p is subtracted from p. The hydrodymanic wave length is $2\pi/k_0 = 2.36$ corresponds in the figures with twice the length of the vorticity blobs. u is discontinuous across $y = 0, x > 0$, whereas v and p are continuous everywhere (outside the source). The velocity fields are confined to the neighbourhoods of source and trailing vortices. Since $\omega \neq \sigma$, the pressure diverges logarithmically.

3.3. *Interpretation for compressible duct flow*

For a comparison with the three-dimensional acoustic problem of a cylindrical duct of radius a, mean flow of Mach number M and boundary layer thickness ah (as considered by Brambley *et al.* (2011b)), we note that in the shear layer we have (in dimensionless form)

$$U(r) = Mh^{-1}(1-r) = Mh^{-1}(1-r_0) + Mh^{-1}(r_0 - r)$$

which is equivalent to the 2D problem if we identify $y = a(r_0 - r)$, $U_0 = c_0 M(1-r_0)/h$, and $\sigma = c_0 M/ah$ and $\omega := \omega c_0/a$, such that the dimensionless duct equivalent of k_0 is

$$k_0 := k_0 a = \frac{a\omega}{U_0} = \frac{\omega h}{M(1-r_0)}.$$

Exactly the same trailing vorticity wave number k_0 is found in the acoustic duct problem as in the present 2D incompressible problem. In the next section we will show that this analogy extends to the configuration where the source is positioned near an impedance wall. We will show that the surface waves excited in the incompressible problem have a clear and strict counterpart among the modes of the acoustic duct problem.

4. A time-harmonic point mass source in linear shear over an impedance wall

4.1. *The soft wall*

Consider the same equations (3.2) as before, but now in a region $y \in [0, \infty)$, with a source at $y = y_0$, and a wall of impedance $Z_w = \rho_0 \zeta$ at $y = 0$ where U vanishes. We have, with $\Omega = \omega - kU$, $U(y) = U_0 + \sigma(y - y_0) = \sigma y$, $U_0 = \sigma y_0$, $\Omega_0 = \omega - kU_0$, $k_0 = \omega/U_0$ and $\tilde{p}(0) = -\rho_0 \zeta \tilde{v}(0)$ at $y = 0$, the same incompressible Pridmore-Brown equation (3.3) and far field conditions as for the free field problem, but now with boundary condition

$$\mathrm{i}\omega \tilde{p}(0) = \zeta \tilde{p}'(0).$$

Note that ζ has the dimension of velocity. Similarly to the free field configuration, the Fourier-transformed solution can be constructed and is found to be

$$\tilde{p} = \frac{\mathrm{i}\rho_0 S}{2|k|\Omega_0}\,\mathrm{e}^{-|k|y_> + |k|y_<}\big(\Omega_> - \mathrm{sign}(\mathrm{Re}\,k)\sigma\big)\big(\Omega_< + \mathrm{sign}(\mathrm{Re}\,k)\sigma\big)$$
$$+\, \frac{\mathrm{i}\rho_0 S}{2|k|\Omega_0}\,\mathrm{e}^{-|k|y_> - |k|y_<}\big(\Omega_> - \mathrm{sign}(\mathrm{Re}\,k)\sigma\big)\big(\Omega_< - \mathrm{sign}(\mathrm{Re}\,k)\sigma\big)\frac{\mathrm{i}k\zeta + \sigma + \mathrm{sign}(\mathrm{Re}\,k)\omega}{\mathrm{i}k\zeta + \sigma - \mathrm{sign}(\mathrm{Re}\,k)\omega}$$

where $y_< = \min(y, y_0)$, $y_> = \max(y, y_0)$ and $\Omega_{<>} = \Omega(y_{<>})$. We distinguish the incident and reflected part:

$$\tilde{p} = \tilde{p}_{in} + \tilde{p}_{ref}$$
$$\tilde{p}_{in} = \frac{\mathrm{i}\rho_0 S}{2|k|\Omega_0}\,\mathrm{e}^{-|k||y-y_0|}(\Omega\Omega_0 - \sigma^2|k||y-y_0| - \sigma^2)$$
$$\tilde{p}_{ref} = \frac{\mathrm{i}\rho_0 S}{2|k|\Omega_0}\,\mathrm{e}^{-|k|(y+y_0)}(\Omega - \mathrm{sign}(\mathrm{Re}\,k)\sigma)(\Omega_0 - \mathrm{sign}(\mathrm{Re}\,k)\sigma)\frac{\mathrm{i}k\zeta + \sigma + \mathrm{sign}(\mathrm{Re}\,k)\omega}{\mathrm{i}k\zeta + \sigma - \mathrm{sign}(\mathrm{Re}\,k)\omega}$$

Again, the physical field in the x, y-domain is obtained by inverse Fourier transformation,

$$p(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{p}(y, k)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k = p_{in} + p_{ref}$$

with a pole at $k = k_0$ (the vorticity shed from the source), and possibly at one or two locations $k = k_s$ given by the dispersion relation for surface wave-like modes

$$k_s = \mathrm{i}\zeta^{-1}\left(\sigma - \mathrm{sign}(\mathrm{Re}\,k_s)\omega\right). \tag{4.1}$$

In particular, assuming that $\sigma > 0$ and $\omega > 0$, and noting that $\mathrm{Re}\,\zeta > 0$ on physical grounds, we may distinguish the following cases (see figure 2)

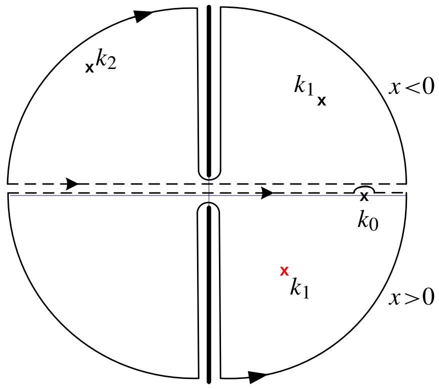


FIGURE 2. Complex k-plane with possible positions of poles, branch cuts of $|k|$, and original (- - -) and deformed (—) Fourier inversion contours for $x < 0$ and $x > 0$.

Case i.	$\mathrm{Im}\,\zeta > 0,\ \sigma > \omega:$	$k_s = k_1 \in$	1st quadrant
Case ii.	$\mathrm{Im}\,\zeta > 0,\ \sigma \leqslant \omega:$	no k_s present	
Case iii.	$\mathrm{Im}\,\zeta < 0,\ \sigma \geqslant \omega:$	$k_s = k_2 \in$	2nd quadrant
Case iv.	$\mathrm{Im}\,\zeta < 0,\ \sigma < \omega:$	$k_s = k_1 \in$	4th quadrant
		$k_s = k_2 \in$	2nd quadrant

where

$$k_1 = \mathrm{i}\zeta^{-1}(\sigma - \omega), \quad k_2 = \mathrm{i}\zeta^{-1}(\sigma + \omega).$$

If $\mathrm{Im}(\zeta) = 0$, the k_s poles are located just on the imaginary axis, i.e. on the branch cut of $|k|$. In that case we have to take the limit $\mathrm{Im}(\zeta) \to 0$ from above or below, with either limit giving the same result. If $\omega = \sigma$, $k_1 = 0$ while there will be no contribution from this pole.

These modes are evidently the incompressible limit of the acoustic surface waves (Rienstra (2003)) that exist for $\mathrm{Im}\,\zeta < 0$ and no flow and are given by

$$k = \pm\mathrm{i}\omega\sqrt{\zeta^{-2} - c_0^{-2}} \simeq \pm\mathrm{i}\omega\zeta^{-1}.$$

There is no such clear relation, however, with the incompressible limit of the surface waves along an impedance wall in a uniform mean flow with Ingard–Myers condition, given by Rienstra (2003). This is indeed not to be expected as we have an infinite shear layer in one case against a vanishing boundary layer in the other. If we rewrite equation (12) of Rienstra (2003) into the present notation (and correct a typo), we obtain the dispersion relation

$$(k_\infty - k)^2 - \mathrm{i}(\zeta/U_\infty)k_\infty|k| = 0,$$

where $k_\infty = \omega/U_\infty$ and U_∞ is the uniform mean flow velocity. This equation has 0, 2 or 4 solutions (one in each quadrant) depending on $\mathrm{Im}\,\zeta/U_\infty$ being > 2, $\leqslant 2$, and $\leqslant -2$ respectively. This is to be compared with the 0, 1 or 2 solutions, depending on the signs of $\mathrm{Im}\,\zeta$ and $\sigma - \omega$, for the present shear flow case.

Because of the presence of the mean flow, it is not immediately clear if the k_s-modes are stable. However, a Briggs-Bers stability analysis (Briggs (1964), Bers (1983)) shows that any k_s is a stable mode. In fact, we will show that, with $\zeta = \zeta(\omega)$, and $\omega = \omega(k)$ defined by dispersion relation (4.1), $\mathrm{Im}\,\omega$ is bounded from below by zero as a function of real k. Indeed, if we have $k \in \mathbb{R}$, then

$$\operatorname{Im}\omega = |k| \operatorname{Re}\zeta(\omega).$$

For a passive liner with $\operatorname{Re}\zeta > 0$ for real ω, this shows that $\operatorname{Im}\omega = 0$ only if $k = 0$. Under reasonable assumptions of smoothness of $\zeta(\omega)$, $\operatorname{Im}\omega(k)$ is continuous and hence can only change sign once, namely at $k = 0$. However, it does not change sign, for the following reason. When $|\zeta(\omega)| \geqslant O(\omega)$ for $\omega \to \infty$ (a reasonable assumption if the impedance involves inertia effects), ζ must vanish for large k, and so $\lim_{k\to\pm\infty}\zeta(\omega) = 0$. Because of causality (Rienstra (2006)), $1/\zeta$ must be analytic in $\operatorname{Im}\omega < 0$ and so any zero of ζ has a positive imaginary part. So $\lim_{k\to\pm\infty}\operatorname{Im}\omega(k)$ is always positive, and in particular $\operatorname{Im}\omega$ is positive on either side of $k = 0$ and therefore does not change sign. Hence, $\min_{k\in\mathbb{R}}(\operatorname{Im}\omega) = 0$. Since this minimum is not negative, the modes are not unstable.

We continue with our construction of an explicit expression for p by noting that p_{in} is the same as for the free field, with y replaced by $y - y_0$, and we denote this free field pressure by p_f (with a similar notation for the velocities). The reflected field is a contribution of the k_0 pole, any k_s poles present, and the branch cut integrals. The contribution from k_0 is only present downstream ($x > 0$). If we close the integral via $-\mathrm{i}\infty$ we capture the k_0-residue of p_{ref}

$$-\tfrac{1}{2}\rho_0 S k_0 \alpha(\alpha U_0 - U_0 + \sigma y)\frac{k_0 - k_2}{k_0 - k_1}\,\mathrm{e}^{-\mathrm{i}k_0\bar{z}_+}\,H(x) \tag{4.2}$$

(where $z_\pm = x + \mathrm{i}(y \pm y_0)$) representing something like the image field of the shed vorticity. For the contributions from k_s poles we have to consider different cases, according to the possible positions of the k_s poles (see below).

Folding the integration contour around the branch cuts (see figure 2), we obtain integrals of the following type (derived in the same way as for (3.11), and with the branch cuts of E still to be determined)

$$\begin{aligned}\int_0^\infty \frac{(\lambda+p_1)(\lambda+p_2)(\lambda+p_3)}{\lambda(\lambda-\mathrm{i}q_1)(\lambda-\mathrm{i}q_2)}\,\mathrm{e}^{-\lambda z}\,\mathrm{d}\lambda &= \frac{1}{z} + \frac{p_1p_2p_3}{q_1q_2}(\gamma+\log z)\\ -\frac{(p_1+\mathrm{i}q_1)(p_2+\mathrm{i}q_1)(p_3+\mathrm{i}q_1)}{q_1(q_1-q_2)}\,\mathrm{E}(q_1,z) &- \frac{(p_1+\mathrm{i}q_2)(p_2+\mathrm{i}q_2)(p_3+\mathrm{i}q_2)}{q_2(q_2-q_1)}\,\mathrm{E}(q_2,z).\end{aligned} \tag{4.3}$$

Altogether we can construct for the various cases explicit results, depending on the locations of the k_s poles. In general the pressure looks like

$$\begin{aligned}p(x,y) = p_f(x,y-y_0) &- \frac{\rho_0 S}{2\pi}\sigma y\frac{x}{x^2+(y+y_0)^2} - \frac{\mathrm{i}\rho_0 S\sigma}{2\pi}(\alpha-\alpha^{-1})\Big[\gamma+\tfrac{1}{2}\log\big(x^2+(y+y_0)^2\big)\Big]\\ -\frac{\mathrm{i}\rho_0 S k_0\alpha}{4\pi}\Big[(\alpha U_0+U_0-\sigma y)&\frac{k_0-k_1}{k_0-k_2}E(k_0,z_+)+(\alpha U_0-U_0+\sigma y)\frac{k_0-k_2}{k_0-k_1}\Big(E(k_0,\bar{z}_+)-2\pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}k_0\bar{z}_+}\,H(x)\Big)\Big]\\ +\frac{\rho_0 S k_0}{2\pi\zeta}\left[\frac{k_1(U_0-\mathrm{i}\zeta)}{k_0-k_1}\right.&\left.(\sigma y-\mathrm{i}\zeta)\big(E(k_1,\bar{z}_+)-C_1\big) + \frac{k_2(U_0+\mathrm{i}\zeta)}{k_0-k_2}(\sigma y+\mathrm{i}\zeta)\big(E(k_2,z_+)-C_2\big)\right]\end{aligned} \tag{4.4}$$

where C_1 and C_2 relate to the possible contributions of the poles k_1 and k_2. If $\alpha = 1$, the formula simplifies since the log-term and the factor of k_1 vanish.

Using similar reasoning, this time without divergent integrals, we obtain from

$$\tilde{v}_{ref} = \tfrac{1}{2} S\,\mathrm{e}^{-|k|(y+y_0)}\Big(1 - \mathrm{sign}(\mathrm{Re}\,k)\frac{\sigma}{\Omega_0}\Big)\frac{\mathrm{i}k\zeta + \sigma + \mathrm{sign}(\mathrm{Re}\,k)\omega}{\mathrm{i}k\zeta + \sigma - \mathrm{sign}(\mathrm{Re}\,k)\omega}$$

$$\tilde{u}_{ref} = \tfrac{1}{2}\mathrm{i} S\,\mathrm{e}^{-|k|(y+y_0)}\Big(\mathrm{sign}(\mathrm{Re}\,k) - \frac{\sigma}{\Omega_0}\Big)\frac{\mathrm{i}k\zeta + \sigma + \mathrm{sign}(\mathrm{Re}\,k)\omega}{\mathrm{i}k\zeta + \sigma - \mathrm{sign}(\mathrm{Re}\,k)\omega}$$

the velocities

$$\begin{aligned} v(x,y) &= v_f(x, y-y_0) + \\ \frac{S}{2\pi}\frac{y+y_0}{x^2+(y+y_0)^2} &+ \frac{Sk_0\alpha}{4\pi}\left[\frac{k_0-k_1}{k_0-k_2}E(k_0,z_+) + \frac{k_0-k_2}{k_0-k_1}\Big(E(k_0,\bar{z}_+) - 2\pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}k_0\bar{z}_+}H(x)\Big)\right] \\ &+ \frac{\mathrm{i}Sk_0}{2\pi\zeta}\left[\frac{k_1(U_0-\mathrm{i}\zeta)}{k_0-k_1}\Big(E(k_1,\bar{z}_+) - C_1\Big) + \frac{k_2(U_0+\mathrm{i}\zeta)}{k_0-k_2}\Big(E(k_2,z_+) - C_2\Big)\right], \end{aligned} \tag{4.5}$$

$$\begin{aligned} u(x,y) &= u_f(x, y-y_0) + \\ \frac{S}{2\pi}\frac{x}{x^2+(y+y_0)^2} &- \frac{\mathrm{i}Sk_0\alpha}{4\pi}\left[\frac{k_0-k_1}{k_0-k_2}E(k_0,z_+) - \frac{k_0-k_2}{k_0-k_1}\Big(E(k_0,\bar{z}_+) - 2\pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}k_0\bar{z}_+}H(x)\Big)\right] \\ &- \frac{Sk_0}{2\pi\zeta}\left[\frac{k_1(U_0-\mathrm{i}\zeta)}{k_0-k_1}\Big(E(k_1,\bar{z}_+) - C_1\Big) - \frac{k_2(U_0+\mathrm{i}\zeta)}{k_0-k_2}\Big(E(k_2,z_+) - C_2\Big)\right]. \end{aligned} \tag{4.6}$$

The contributions C_1 and C_2 are as follows.

Case i (1 pole). $k_s = k_1$ is found in the upper half plane and therefore contributes upstream.

$$C_1 = -2\pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}k_1\bar{z}_+}H(-x), \quad C_2 = 0. \tag{4.7a}$$

Case ii (no pole). No k_s pole present, so

$$C_1 = 0, \quad C_2 = 0. \tag{4.7b}$$

Case iii (1 pole). $k_s = k_2$ is found in the upper half plane and contributes upstream.

$$C_1 = 0, \quad C_2 = 2\pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}k_2 z_+}H(-x). \tag{4.7c}$$

Case iv (2 poles). one $k_s = k_1$ is now found in the lower half plane and contributes downstream, while a second $k_s = k_2$, and therefore C_2, is the same as in case iii above. We have

$$C_1 = 2\pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}k_1\bar{z}_+}H(x), \quad C_2 = 2\pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}k_2 z_+}H(-x). \tag{4.7d}$$

The exponential integral E_1 in the function $E(q,z)$, $q \in \mathbb{C}$, defined in (5.2), does not follow the standard definition anymore. Instead of a branch cut along the negative real axis of the argument, the branch cut is rotated and depends on q in such a way that if $\mathrm{Re}(q) > 0$, the branch cut of $E_1(-\mathrm{i}qz)$ is always mapped along the line $x = 0, y < 0$, and thus for $E_1(-\mathrm{i}q\bar{z})$ is the branch cut located along the line $x = 0, y > 0$; if $\mathrm{Re}(q) < 0$ it is the other way round (see Appendix for more details). This definition only differs from the standard one if q is not both real and positive, and therefore agrees with $E(k_0, z)$ in (3.11).

As for the free field problem, the branch cuts of the E functions now compensate for the jumps of the Heaviside functions. These are in no way physical but artefacts of the contour being closed via the lower (if $x > 0$) or upper (if $x < 0$) complex half plane. The resulting fields are otherwise smooth and continuous.

4.2. *The hard wall limit*

A special case of interest is the hard wall, i.e. $\zeta \to \infty$. This limit is relatively easily found for the velocities. They show a certain symmetry about $y = 0$

$$\begin{aligned} v_{HW}(x,y) = {}& \frac{S}{2\pi}\frac{y-y_0}{x^2+(y-y_0)^2} - \frac{Sk_0\alpha}{4\pi}\big[E(k_0,z_-)+E(k_0,\bar{z}_-)-2\pi\mathrm{i}H(x)\,\mathrm{e}^{-\mathrm{i}k_0x-k_0|y-y_0|}\big] \\ & -\left(\frac{S}{2\pi}\frac{-(y+y_0)}{x^2+(y+y_0)^2} - \frac{Sk_0\alpha}{4\pi}\big[E(k_0,\bar{z}_+)+E(k_0,z_+)-2\pi\mathrm{i}H(x)\,\mathrm{e}^{-\mathrm{i}k_0x-k_0(y+y_0)}\big]\right), \end{aligned} \tag{4.8}$$

$$\begin{aligned} u_{HW}(x,y) = {}& \frac{S}{2\pi}\frac{x}{x^2+(y-y_0)^2} + \frac{S}{2\pi}\frac{x}{x^2+(y+y_0)^2} \\ & + \frac{\mathrm{i}Sk_0\alpha}{4\pi}\big[E(k_0,z_-)-E(k_0,\bar{z}_-)+2\pi\mathrm{i}H(x)\,\mathrm{sign}(y-y_0)\,\mathrm{e}^{-\mathrm{i}k_0x-k_0|y-y_0|}\big] \\ & + \frac{\mathrm{i}Sk_0\alpha}{4\pi}\big[E(k_0,\bar{z}_+)-E(k_0,z_+)+2\pi\mathrm{i}H(x)\,\mathrm{sign}(-(y+y_0))\,\mathrm{e}^{-\mathrm{i}k_0x-k_0(y+y_0)}\big]. \end{aligned} \tag{4.9}$$

As a result, the expressions can be written in terms of the free field velocities as follows.

$$\begin{aligned} v_{HW}(x,y) &= v_f(x,y-y_0) - v_f(x,-(y+y_0)), \\ u_{HW}(x,y) &= u_f(x,y-y_0) + u_f(x,-(y+y_0)). \end{aligned} \tag{4.10}$$

The hard wall limit for pressure, on the other hand, is more subtle than may be expected, because the inherent undetermined additional constant depends on ζ in the general soft wall case. So any hard-wall limit will not (immediately) agree with a corresponding case of finite but large ζ.

If we take directly the limit of (4.4) and ignore the diverging $\log\zeta$-terms, we find

$$\begin{aligned} p_{HW}(x,y) = {}& p_f(x,y-y_0) - p_f(x,-(y+y_0)) - \frac{\mathrm{i}\rho_0 S\sigma}{2\pi}\Big[E(k_0,z_+)-E(k_0,\bar{z}_+) \\ & + 2\pi\mathrm{i}H(x)\,\mathrm{e}^{-\mathrm{i}k_0\bar{z}_+} + (1+\alpha^{-1})\,\mathrm{Log}(-\mathrm{i}z_+) - (1-\alpha^{-1})\,\mathrm{Log}(\mathrm{i}\bar{z}_+)\Big] \end{aligned} \tag{4.11}$$

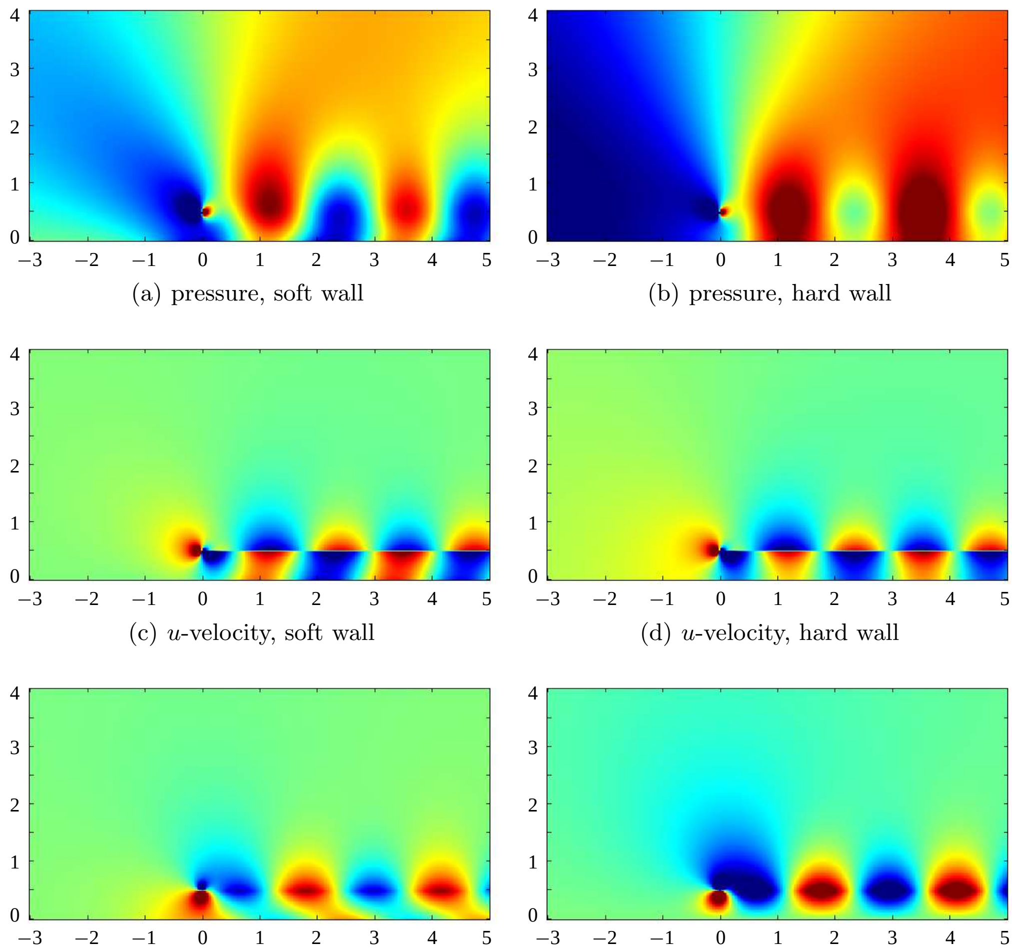
where Log denotes the principal value logarithm (see eq. 5.2). p_{HW} is, remarkably, not similar to the corresponding expressions (4.10) for the velocities.

4.3. *Examples*

In figures 3 and 4 graphical representations are given for pressure and velocity fields along soft and hard walls for a number of typical cases: $\omega = 8$ combined with $\sigma = 6$ and $\sigma = 10$, and hard walls compared with soft walls of $\zeta = 4 - 2\mathrm{i}$, corresponding with a case iii and a case iv. This value of ζ is selected to be of the order of magnitude of U_0. The variation in σ has a significant effect. However, taking $\zeta = 4 + 2\mathrm{i}$, corresponding with case i and case ii situations, does not give significantly different results and is therefore not shown here. Again the effect of the undetermined constant in p is removed by subtracting its plot-domain averaged value.

The parameters chosen for figure 3 are $\omega = 8$, $y_0 = 0.5$, $\sigma = 6$, and hence $U_0 = 3$, $k_0 = 2.67$, $\alpha = 0.75$, and $k_1 = 0.2 - 0.4\mathrm{i}$ and $k_2 = -1.4 + 2.8\mathrm{i}$ (case iv) for the soft wall. The hydrodymanic wave length is $2\pi/k_0 = 2.36$, just large enough to have some interaction with the wall.

The parameters chosen for figure 4 are $\omega = 8$, $y_0 = 0.5$, $\sigma = 10$, and and hence $U_0 = 5$, $k_0 = 1.6$, $\alpha = 1.25$, and $k_1 = -0.2 + 0.4\mathrm{i}$ and $k_2 = -1.8 + 3.6\mathrm{i}$ (case iii) for the soft wall. The hydrodymanic wave length $2\pi/k_0 = 3.93$ is now large compared to the wall distance



(a) pressure, soft wall (b) pressure, hard wall

(c) u-velocity, soft wall (d) u-velocity, hard wall

(e) v-velocity, soft wall (f) v-velocity, hard wall

FIGURE 3. $\omega = 8$, $\sigma = 6$, $y_0 = 0.5$, giving $U_0 = 3$, $k_0 = 2.67$, $\alpha = 0.75$. For the soft wall: $\zeta = 4 - 2\mathrm{i}$, $k_1 = 0.2 - 0.4\mathrm{i}$, $k_2 = -1.4 + 2.8\mathrm{i}$ (case iv)

$y_0 = 0.5$. This results into a strong interaction of the shed vorticity field with the wall, especially for the velocities.

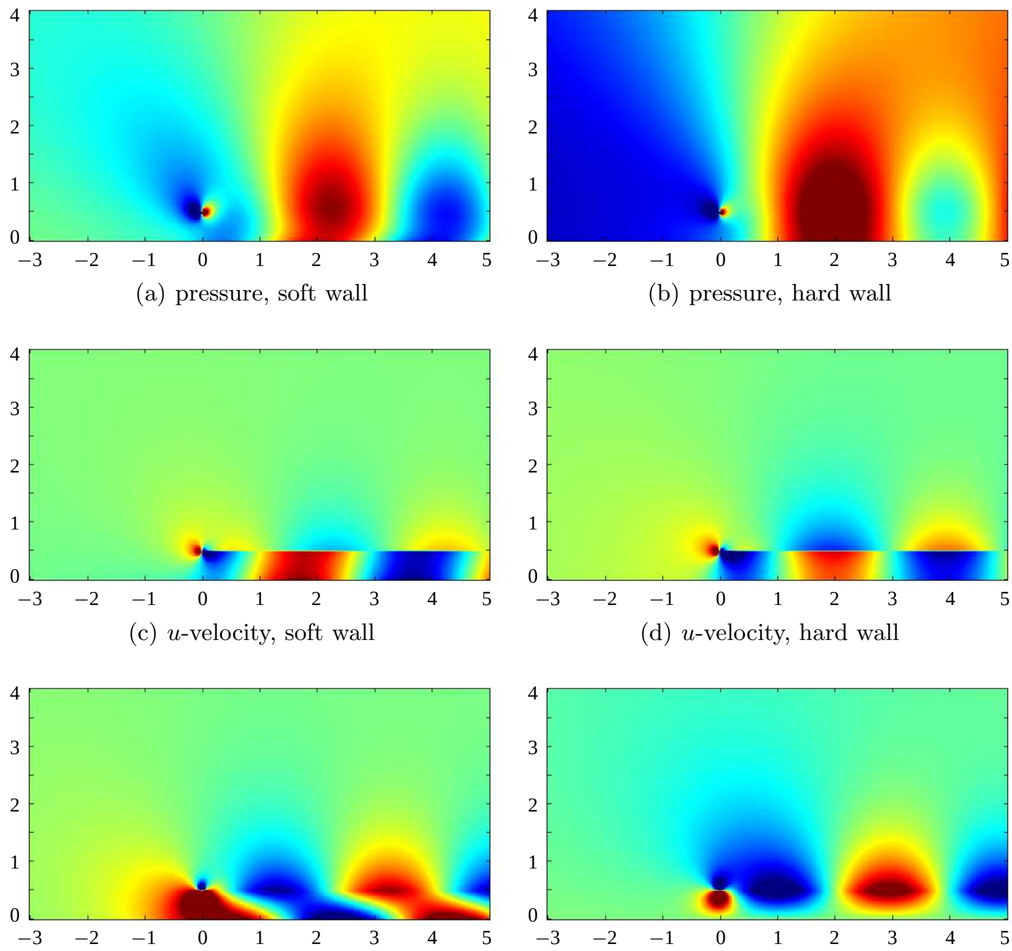
4.4. *Interpretation for acoustic duct modes with lined walls*

In order to compare with the acoustic problem of a lined flow duct, we note that $\rho_0\zeta = \rho_0 c_0 Z$ such that the dimensionless duct equivalents of k_s are

$$k_s := k_s a = \frac{\mathrm{i}}{Z}\left(\frac{M}{h} \pm \omega\right).$$

These correspond indeed to two of the compressible surface modes, as is clearly seen in figure 5. We use the notation and geometry of Brambley *et al.* (2011b), i.e. a cylindrical duct with linear-then-constant mean flow

$$U(r) = \begin{cases} M, & 0 \leqslant r \leqslant 1 - h, \\ M(1-r)/h, & 1 - h \leqslant r \leqslant 1, \end{cases}$$



(a) pressure, soft wall

(b) pressure, hard wall

(c) u-velocity, soft wall

(d) u-velocity, hard wall

(e) v-velocity, soft wall

(f) v-velocity, hard wall

FIGURE 4. $\omega = 8$, $\sigma = 10$, $y_0 = 0.5$, giving $U_0 = 5$, $k_0 = 1.6$, $\alpha = 1.25$.
For the soft wall: $\zeta = 4 - 2\mathrm{i}$, $k_1 = -0.2 + 0.4\mathrm{i}$, $k_2 = -1.8 + 3.6\mathrm{i}$ (case iii)

with M the mean flow Mach number.

5. Conclusions

The analytically exact and explicit solutions for the problems of a time-harmonic point mass source in incompressible inviscid two-dimensional linear shear mean flow are derived for the free field situation and for a semi-infinite space with the mean flow directed parallel to and vanishing at an impedance wall. Both problems were motivated by the problem of an acoustic source in a shear flow (for example a boundary layer) along a lined wall.

Both solutions were obtained by Fourier transformation in the mean flow direction x, which leads to equations in the cross wise direction y that are solvable exactly (Rayleigh (1945), Drazin & Reid (2004)). The inverse transform of the pressure, however, leads to divergent integrals because of the well known logarithmic behaviour of the far field of a point source in 2D incompressible flow. If the medium were compressible, the incompressible far field would have changed into a decaying compressible outer field. The present

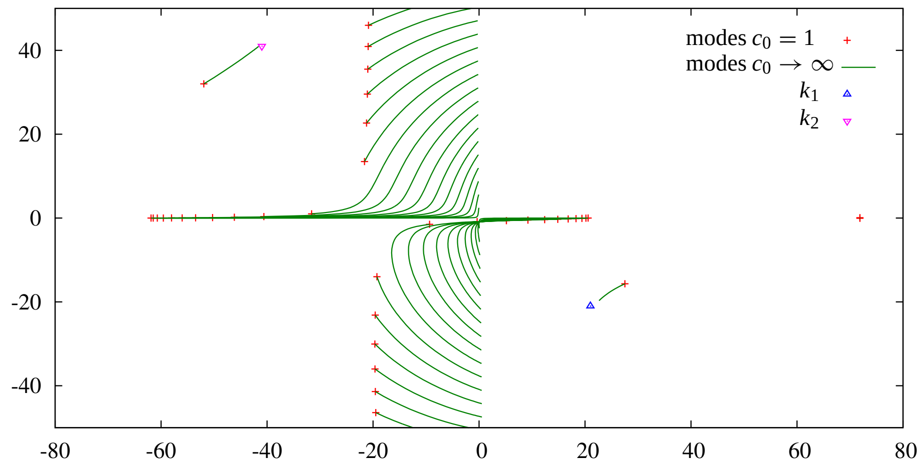


FIGURE 5. Tracking the k_1 and k_2 surface modes in a 3D duct, for a medium turning from fully compressible to incompressible. A case iv parameter choice is shown, with $\omega = 31$, $m = 0$, $U_0 = 0.5$, $h = 0.05$, $Z = 0.5 - 0.5\mathrm{i}$ (scaled on ρ_0).

problem of a divergent integral is not insurmountable and can be cured by interpreting the integrals in a generalised sense.

The dominating feature in the solutions found is a non-decaying train of vortices, shed from the point source (possibly comparable with a von Kármán vortex street). This trailing vorticity field is essentially due to the mean flow shear. In a uniform mean flow, a point source would not generate vorticity, as shown in (2.7). Generating vorticity in uniform mean flow would require at least a point force.

Since the pressure has no natural boundary conditions at infinity, this variable is only determined up to an undetermined constant. This constant may depend on the impedance, so if we apply large but finite impedances, the found pressure may become large in amplitude, and singular in the limit of hard walls. So care is to be taken when the hard wall limit is applied, and in general it may be better to always subtract a mean value in order to keep the pressure levels within reasonable limits.

All this is not the case with the velocity. Here the natural conditions at infinity are a vanishing velocity, such that the method of Fourier transformation automatically sifts out the required solution.

In a compressible context, this work identifies the k_0 pole found by Brambley *et al.* (2011b) as shed vorticity from the point source, as was speculated. However, this work also shows that the possibly related surface modes k_+ and k_- found by Brambley *et al.* (2011b) are not present here and therefore do not have incompressible infinite-shear counterparts, suggesting that the physics causing them may be significantly different from the physics causing the k_0 pole despite their similar locations in the k-plane. As shown in figure 5, the two surface modes that do occur here correspond to another two of the possible six compressible surface modes for compressible shear flow over a lining (Brambley (2011)).

Acknowledgement

M. Darau's contribution was part of PhD work in a cooperation between TU Eindhoven (Netherlands) and the West University of Timişoara (Romania), supervised by professors Robert R.M. Mattheij and Stefan Balint. She acknowledges the support by a grant of the

Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0171. E.J. Brambley was supported by a Research Fellowship at Gonville & Caius College, Cambridge.

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Appendix: The exponential integral

An important function in the foregoing analysis is the function $E(q, z)$, closely related to the exponential integral E_1 (see Abramowitz & Stegun (1965), equation 5.1.1). For $q, z \in \mathbb{C}$ we have

$$\begin{aligned} E(q,z) &= \mathrm{e}^{-\mathrm{i}qz}\, E_1(-\mathrm{i}qz), \\ E_1(z) &= \int_z^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t = -\gamma - \log z + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^k}{k\,k!}, \end{aligned} \tag{5.1}$$

where $\gamma = 0.5772156649\ldots$ is Euler's constant. The variable q corresponds here to complex wave numbers (k_0 and k_s), while $z = x + \mathrm{i}y$ relates to the physical (x, y)-space. As is clear from the series representation, $E_1(z)$ has a logarithmic singularity, for which the standard definition is to use the principal value, Log, with $\mathrm{Log}(1) = 0$ and a branch cut along the negative real axis. However, with our applications this choice would result in q-dependent branch cuts in the x, y-domain and is therefore not convenient.

The branch cut discontinuity is the counterpart of the discontinuity at the line $x = 0$ due to the upward or downward closure of the Fourier integral contour for $x < 0$ or $x > 0$ respectively. Therefore, a most natural location for the branch cut is the imaginary axis.

The choice used here is such that if $\mathrm{Re}\, q > 0$ the branch cut of $E(q, z)$ is along the line $x = 0, y < 0$ and thus for $E(q, \bar{z})$ along the line $x = 0, y > 0$. If $\mathrm{Re}\, q < 0$ it is the opposite: the branch cut of $E(q, z)$ is then taken along the line $x = 0, y > 0$. This is most easily obtained by the logarithm

$$\log(-\mathrm{i}qz) \stackrel{\text{def}}{=} \mathrm{Log}\Big(-\mathrm{i}z\frac{q}{|q|}\Big) + \mathrm{Log}(|q|)$$

with the principal value Log and $|q|$ as defined in (3.5). So we define, with *this* log and E_1 the *standard* exponential integral, our function E as

$$E(q,z) \stackrel{\text{def}}{=} \mathrm{e}^{-\mathrm{i}qz}\left(E_1(-\mathrm{i}qz) + \mathrm{Log}(-\mathrm{i}qz) - \mathrm{Log}\Big(-\mathrm{i}z\frac{q}{|q|}\Big) - \mathrm{Log}(|q|)\right). \tag{5.2}$$