Cosmology Part III

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Contents

LESSON 1

Overview

Modern cosmology rests on four pillars: and expanding universe, Large Scale Structures, Cosmic Microwave Background and inflation. The plan of 24 hours is as follows

- 6h: A smooth expanding universe, FLRW, distances, constituents and thermal history
- 2h: Inflationary background
- 4h: Cosmological perturbation theory (CPT)
- 4h: Inflationary perturbations
- 4h: Cosmic Microwave Background (CMB)
- 4h: Large Scale Structures (LSS)

I use the shorthand notations: D for Dodelson's Modern Cosmology book [13], W for Weinberg's Cosmology book [51]. For example D 3 is Chapter 3 of Donelson's book, while W appB is appendix B of Weinberg's book.

Check For Understanding (cfu) This a question that requires a short back-of-the-envelope calculation.

Check for understanding (CFU) This is a more conceptual or intuitive question that requires reasoning but not necessary a calculation.

Notation, units and conventions I use units in which $\hbar = c = k_b = 1$. Therefore energy is temperature and inverse time or inverse length. On the other hand, I will try to keep the reduced Planck mass explicit, $M_{\rm Pl} = (8\pi G_N)^{-1/2}$. Beware that some authors use $M_{\rm Pl}$ to indicate the "full" Planck mass $G_N^{-1/2} \simeq 1.2 \times 10^{19}$GeV. The necessary conversion factors can be added using dimensional analysis and

$$c = 3 \times 10^8 \frac{\rm m}{\rm sec}\,, \quad {\rm pc} = 3.2\,{\rm lightyears}\,, \quad {\rm year} = \pi \times 10^7\,{\rm sec}\,, \tag{1.1}$$

$$\hbar c = 0.2\,{\rm eV}\,\mu{\rm m}\,, \quad M_{\rm Pl} \simeq 2.4 \times 10^{18}{\rm GeV}\,. \tag{1.2}$$

I use the mostly plus signature $(-,+,+,+)$. Latin indices indicate space, $i, j, \dots = \{1,2,3\}$, while greek indices run over spacetime, $\mu, \nu, \dots = \{0,1,2,3\}$. 3D vectors are in boldface, e.g. $\mathbf{k}$ and $\mathbf{x}$. Unless otherwise specified, all tensors are expressed in terms of the FLRW coordinates

$$ds^2 = -dt^2 + a^2 dx^2\,. \tag{1.3}$$

. Standard derivatives are represented with a comme and covariant derivatives with a semi-column

$$T^{\cdots}_{\cdots,\mu} \equiv \partial_\mu T^{\cdots}_{\cdots}\,, \quad T^{\cdots}_{\cdots;\mu} \equiv \nabla_\mu T^{\cdots}_{\cdots}\,. \tag{1.4}$$

Symmetrization and anti-symmetrization of a pair of indices is indicated with $(\dots)$ and $[\dots]$ respectively and is defined to have weight 1

$$A_{(\mu\nu)} \equiv \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right)\,, \quad A_{[\mu\nu]} \equiv \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right)\,. \tag{1.5}$$

My convention for the Fourier transform are

$$F(\mathbf{x}) = \int_{\mathbf{k}} \tilde{F}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad F(\mathbf{k}) = \int_{\mathbf{x}} \tilde{F}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \quad \text{with} \quad \int_{\mathbf{k}} \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,, \quad \int_{\mathbf{x}} \equiv \int d^3\mathbf{x}\,. \tag{1.6}$$

There are surprisingly many conventions for the name of variables in perturbation theory. In particular, Newtonian gauge is written as

$$\begin{aligned} ds^2 &\equiv -\left(1+2\Psi_D\right) dt^2 + a^2\left(1+2\Phi_D\right) dx^i dx^j \delta_{ij} && (1.7)\\ &\equiv -\left(1+2\Phi_W\right) dt^2 + a^2\left(1-2\Psi_W\right) dx^i dx^j \delta_{ij} && (1.8)\end{aligned}$$

in Dodelson's (D) or Weinberg's (W) notations. The conversion is $\Psi_D = \Phi_W$ and $\Phi_D = -\Psi_W$. In these notes, I use Dodelson's notation everywhere except in Les. 10 and Les. 10.7 where I keep the label W explicit. In all other lessons I drop the D to simplify the notation.

LESSON 2

General Relativity in a nutshell

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In this lesson, I give a lightning review of the results in General Relativity (GR) that we will need in this class and I set my notation. The reader familiar with GR and keen to start with cosmology can skip this lesson move directly to FLRW spacetimes. In the following, I discuss the equivalence principle, geodesic equation, conservation of energy-momentum tensor and conserved charge densities.

2.1 General Relativity

In GR the well-known metric of Minkowski spacetime $\eta_{\mu\nu}$ is promoted to a dynamical metric field $g_{\mu\nu}$. The dynamics of $g_{\mu\nu}$ is determined by distribution of matter. The metric is not determined univocally, but only up to a choice of coordinates, which change the metric in a specific way (somewhat similar to gauge theories). The laws of nature formulated in GR are not only valid in every intertial frame, but in any frame whatsoever. The phenomenon of gravitation is described by postulating that object move along geodesics (shortest paths) of the metric. Additional dynamics due to non-gravitational forces is generalized from the usual Minkoskian expression to account for the coordinate covariance of the theory.

One can derive all General Relativity (GR) from two principles:

- The principle of equivalence of mass and inertia, a.k.a. the *equivalence principle*: free falling observers do not feel the effects of gravitation. Formally, in an open set around any spacetime point I can choose the *locally inertial frame* (LIF), namely coordinates such that the *metric tensor* is approximately Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu}\,, \quad \text{and} \quad \partial_\gamma g_{\mu\nu} \equiv g_{\mu\nu,\gamma} = 0\,. \tag{2.1}$$

- The principle of *general covariance*[1]: equations must be invariant in form under a change of coordinates.

Strategy: first, write down the equations governing a (sufficiently small) system in the absence of gravity; second, re-write them in a covariant way. The equation is now valid in the presence of gravity, i.e. in any coordinates.[2]

Clocks, rods and tensors Take two spacetime points separated by an infinitesimal timelike interval. We call this a clock because there is a reference frame in which this is some observer proper time. To go from special to general relativity I just start from the right expression in the LIF and make it generally covariant:

$$dx^\mu dx^\nu \eta_{\mu\nu} \doteq -dT^2 \quad \to \quad dx^\mu dx^\nu g_{\mu\nu} = -dT^2\,, \tag{2.2}$$

where I use the signature $(-,+++)$. Similarly for length contraction consider an infinitesimal spacelike interval, aka a rod,

$$dx^\mu dx^\nu \eta_{\mu\nu} \doteq dL^2 \quad \to \quad dx^\mu dx^\nu g_{\mu\nu} = dL^2\,. \tag{2.3}$$

Clocks and rod are dilated and contracted in the presence of a gravitation field (unlike in special relativity).

Covariant objects A covariant or contravariant *scalar*, *vector* and *tensor* transform under a change of coordinates $x' = x'(x)$ as

$$\boxed{\phi'(x') = \phi(x)\,, \quad v'^{\mu'} = \frac{\partial x'^{\mu'}}{\partial x^\mu} v^\mu\,, \quad g'_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x'^{\mu'}}\frac{\partial x^\nu}{\partial x'^{\nu'}} g_{\mu\nu}}\,. \tag{2.4}$$

[1] cfu: *Be sure to understand the difference between covariance and invariance*

[2] cfu: *Is a free falling elevator a locally inertial frame? (yes) Are we in this room in a LIF? (no) Is the moon in a LIF? (yes as a point particle, no because of small tides) the earth or the sun? (same as the moon)*

A trick to get it right is to put the prime both on the tensor and on the indices. A tensor that is zero in one frame is zero in every frame.[3][4] Normal derivatives do not in general transform as tensors (unless they act on a scalar), and need to be supplemented by a "connection" to transform covariantly. The covariant derivative ∇_μ, indicated also by the label $;\mu$ appended to tensor it acts on, is defined as follows

$$\begin{aligned}
\nabla_\mu A &= A_{;\mu} = \frac{\partial A}{\partial x^\mu} = \partial_\mu A = A_{,\mu} \,, && (2.5)\\
\Box A \equiv \nabla^\mu \nabla_\mu A &= A^{;\mu}_{;\mu} = \frac{1}{\sqrt{-g}}\partial_\mu \left(g^{\mu\nu}\sqrt{-g}\partial_\nu A\right) \,, && (2.6)\\
A^\mu_{;\nu} &= \frac{\partial A^\mu}{\partial x^\nu} + \Gamma^\mu_{\sigma\nu} A^\sigma \,, && (2.7)\\
A_{\mu;\nu} &= \frac{\partial A_\mu}{\partial x^\nu} - \Gamma^\sigma_{\mu\nu} A_\sigma \,, && (2.8)\\
A^\mu_{\sigma;\nu} &= \frac{\partial A^\mu_\sigma}{\partial x^\nu} - \Gamma^m_{\sigma\nu} A^\mu_m + \Gamma^\mu_{m\nu} A^m_\sigma \,, && (2.9)\\
A_{\mu\sigma;\nu} &= \frac{\partial A_{\mu\sigma}}{\partial x^\nu} - \Gamma^\rho_{\mu\nu} A_{\rho\sigma} - \Gamma^\rho_{\sigma\nu} A_{\mu\rho} \,, && (2.10)
\end{aligned}$$

where the Christoffel symbol

$$(2.11)$$

$$\Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\gamma}\left(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma}\right) \,, \tag{2.12}$$

vanishes in the LIF[5]. Note for this form of the geodesic equation to be valid, u must be linearly related to the proper time (for a generic time parameter u and additional term appears in this equation, see e.g. []). It is useful to remember that

$$\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu} \,, \quad \Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho} = g_{\mu\nu,\rho} \,, \quad \Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{g}}\partial_\lambda \sqrt{g} \,, \tag{2.13}$$

and that $\Gamma^\mu_{\nu\rho}$ does *not* not transform as a tensor. Under a coordinate transformation $x^\mu \to y^\mu(x)$ one finds

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\mu_{\nu\rho} J^\alpha_\mu J^\nu_\beta J^\rho_\gamma + J^\alpha_\mu \partial_\beta J^\mu_\gamma \,, \quad J^\alpha_\mu \equiv \frac{\partial y^\alpha}{\partial x^\mu} \,. \tag{2.14}$$

Geodesic equation Derivative of dx^μ with respect to some fixed time u, such as the proper time along a trajectory dx^μ is a vector because only dx^μ changes, while u does not since it is uniquely defined by the timeline of dx^μ. But the second derivative is not a vector. We need another non-vector to make a covariant expression. From Newton's law in the LIF we find

$$\frac{d^2 x^\mu}{du^2} \doteq 0 \quad \to \quad \boxed{\frac{d^2 x^\mu}{du^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{du}\frac{dx^\beta}{du} = 0} \,. \tag{2.15}$$

The solution are called geodesics and *maximize* proper time for any timelike path connection two timelike events.

Riemann and Ricci In Euclidean spacetime, initially parallel geodesics, i.e. straight lines, remain forever parallel and never intersect. Conversely, the convergence or divergence of geodesics is a manifestation of the curvature of spacetime. Similarly, covariant derivatives on a curved spacetime do not commute and parallel transport along a closed loop does not leave a vector unchanged. The Riemann tensor quantifies the deviation from flat spacetime expectation

$$[\nabla_\mu, \nabla_\nu] V_\rho = R_{\rho\sigma\mu\nu} V^\sigma \,, \tag{2.16}$$

[3] cfu: *What is the inverse of $g_{\mu\nu}$? ($g^{\mu\nu}$)*

[4] cfu: *What is the metric (tensor)? (An infinitesimal distance, i.e. the norm of tangent vectors, not the distance between points on the manifold. By integrating the norm of the tangent vector to some curve (computed with the metric tensor), we can compute the length of the curve. Define by transforming as a two tensor and being $\eta_{\mu\nu}$ in the LIF.)*

[5] cfu: *Why is it called "geodesic" equation? Because the solution minimizes $\int \sqrt{\partial_u x^\mu \partial_u x_\mu} du$*

where the covariant tensor $R^{\rho}{}_{\sigma\mu\nu}$ is given by[6]

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{[\mu}\Gamma^{\rho}_{\nu]\sigma} + \Gamma^{\rho}_{[\mu\lambda}\Gamma^{\lambda}_{\nu]\sigma}\,, \tag{2.17}$$

with anti-symmetrization defined in (1.5). The follow symmetry properties are useful

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}\,, \qquad\qquad R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} = R_{\sigma\rho\nu\mu}\,, \tag{2.18}$$

$$R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} + R_{\rho\mu\nu\sigma} = 0\,, \tag{2.19}$$

where the latter is also known as first (algebraic) Bianchi identity. The second Bianchi identities are instead a differential relation among the components of the Riemann tensor

$$\nabla_{\lambda}R_{\alpha\beta\mu\nu} + \nabla_{\nu}R_{\alpha\beta\lambda\mu} + \nabla_{\mu}R_{\alpha\beta\nu\lambda} = 0 \tag{2.20}$$

which can be derived from the Jacobi identities for the commutator (2.16) of covariant derivatives (see Sec 7.8 of [6]).

Two well known contractions of Riemann are the Ricci tensor and Ricci scalar[7],

$$R_{\mu\nu} \equiv R_{\rho\mu\rho\nu}\,, \qquad R \equiv g^{\mu\nu}R_{\mu\nu} = R^{\mu}_{\mu}\,. \tag{2.21}$$

Contracting all but one of the indices in the Bianchi equations (2.20) with the metric, one gets a contracted Bianchi identity for the Einstein tensor $G_{\mu\nu}$

$$\nabla^{\mu}G_{\mu\nu} \equiv \nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0\,. \tag{2.22}$$

Box 2.1 Newtonian limit and gravitational time dilation This limit is relevant for the formation of Large Scale Structures (LSS). Consider slowly moving particles, namely $\partial_u x^i \ll \partial_u x^0$ and choose time to be proper time so that $\partial_u x^0 = 1$. Then the geodesic equation is simply

$$\frac{d^2x^i}{du^2} = -\Gamma^i_{00}\,. \tag{2.23}$$

For weak gravity, $g = \eta + h$, we can expand to linear order in h. Assuming slow time dependence wrt the spacial dependence $\partial_0 h \ll \partial_i h$ one finds Newton's law of gravitation:

$$\ddot{x}^i = -\partial_i\phi\,, \tag{2.24}$$

with $g_{00} = -1 - 2\phi$ and ϕ the gravitational potential. This implies that proper time runs slower in the gravitational field of a planet ($\phi < 0$):

$$dT = \sqrt{-dx^{\mu}dx^{\nu}g_{\mu\nu}} = \sqrt{(1+2\phi)dx^0dx^0} \simeq (1+\phi)dx^0\,, \tag{2.25}$$

where dT is the proper time interval and dx^0 is some global time coordinate that we use to compare observers with different values of ϕ.

Einstein Equations and Energy-momentum tensor In GR the metric is dynamical and it's evolution is dictated by the EE's: the matter energy momentum tensor tells spacetime how to bend. If the matter theory is described by an action S, then [8] the energy momentum is given by (the sign depends on conventions)

$$\boxed{T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}}\,. \tag{2.26}$$

If we limit ourselves to only two spacetime derivatives, there is only one covariant expression that reduces to Poisson equation:

$$\partial_i\partial^i\phi \doteq 4\pi G\rho \quad \to \quad \boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} = -M_{\rm Pl}^{-2}T_{\mu\nu}}\,. \tag{2.27}$$

[6] cfu: *How many spacetime derivates acting on the metric appear in* ${}^{\rho}{}_{\sigma\mu\nu}$*?*

[7] CFU: *How do* ${}^{\rho}{}_{\sigma\mu\nu}$*,* $R_{\mu\nu}$ *and* R *change under a constant rescaling of the metric* $g_{\mu\nu\to\lambda g_{\mu\nu}}$*?*

[8] cfu: *How do I know this formula is right? It's covariant and in the comoving LIF* $u^{\mu} = (1,0,0,0)$

As a consequence $T^{\mu\nu}_{;\nu} = 0$. The conservation of energy ($\mu = 0$) and momentum ($\mu = i$) currents is given in GR by[9][10]

$$T^{\mu\nu}_{,\nu} \doteq 0 \quad \rightarrow \quad T^{\mu\nu}_{;\nu} \equiv T^{\mu\nu}_{,\mu} + \Gamma^{\mu}_{\kappa\nu} T^{\kappa\nu} + \Gamma^{\nu}_{\kappa\nu} T^{\kappa\mu} = 0\,. \tag{2.28}$$

EE's can be derived from the Einstein Hilbert action (plus the Gibbons-Hawking-York boundary term which I omit here)

$$\boxed{S = \int d^4x \sqrt{-g} \frac{M_{\rm Pl}^2}{2} (R + \Lambda)}\,, \tag{2.29}$$

where Λ is a *cosmological constant*.

Box 2.2 Conserved currents and charges Symmetries of the law of physics are transformations that commute with the time evolution and generate new solutions from old ones. Mathematically, we represent symmetries by transformations of the (field) variables that leave the action invariant. By Noether theorem, for each such symmetry, there is an associated conserved current $\partial_\mu J^\mu \doteq 0$. The corresponding covariant expression is clearly $\nabla_\mu J^\mu = 0$. But the distinction is actually irrelevant (as long as one is careful with the convention she is using) since for every covariantly conserved current J^μ, with $J^\mu_{;\mu} = 0$, one can define a normally conserved current $\tilde{J}^\mu \equiv \sqrt{-g} J^\mu$, since

$$\tilde{J}^\mu_{,\mu} = \sqrt{-g} J^\mu_{;\mu} = 0\,. \tag{2.30}$$

The conserved charge $\dot{Q} = 0$ is then defined as usual by

$$Q \equiv \int d^3x \tilde{J}^\mu n_\mu = \int d^3x \sqrt{g} J^\mu n_\mu\,, \tag{2.31}$$

where the integral is over some spatial hypersurface defined by the perpendicular vector n^μ. It is always possible and sometimes useful to split a current as $J^\mu = \rho u^\mu$ with a normalised velocity u^μ and a density ρ:

$$\rho \equiv \sqrt{-J_\mu J^\mu}\,, \quad u^\mu \equiv \frac{J^\mu}{\rho} \quad \Rightarrow \quad u^\mu u_\mu = -1\,. \tag{2.32}$$

The density ρ is a density per proper volume, which is transformed into a density per coordinate volume by the $\sqrt{-g}$ factor in (2.30).

Lorentz symmetries are special. They lead to the covariant conservation of the energy momentum tensor $T_{\mu\nu}$, see (2.28), but this cannot in general be used to define a conserved charge because the trick in (2.30) does not work for a two-tensor,

$$0 = \sqrt{-g} T^\mu_{\nu;\mu} = (\sqrt{-g} T^\mu_\nu)_{,\mu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda}\,, \tag{2.33}$$

and we are stuck with the last term. Energy and momentum are globally conserved iff there exist a Killing vector ϵ, which satisfies (3.22), $\epsilon_{(\mu;\nu)} = 0$. Then one can build the covariantly conserved current $J^\mu_\epsilon \equiv T^\mu_\nu \epsilon^\nu$, which has only one index, and proceed as above. In cosmology we will be interested in homogeneous and isotropic spacetimes, with six space-like Killing vectors but no time-like Killing vector. As a consequence we can define some globally conserved momentum and angular momentum, but no globally conserved energy. Intuitively, the cosmological spacetime exchanges energy with any system living on it, injecting and subtracting energy depending on the dynamics.

Fluids A relativistic *perfect fluid* is defined as "as a medium for which at every point there is a locally inertial Cartesian frame of reference, moving with the fluid, in which the fluid appears the same in all directions." (see B.10 of [51]). In the comoving LIF therefore the energy-momentum tensor must be diagonal and isotropic: $T^\mu_\nu \doteq (-\rho, p, p, p)$. By boosting with a velocity u^μ, which is a timelike vector $u_\mu u^\mu = -1$, one finds the covariant form of the energy-momentum tensor

$$\boxed{T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + g^{\mu\nu} p}\,. \tag{2.34}$$

Here the energy density ρ and the pressure p are covariant scalars, while u^μ is a covariant vector. Conversely, if you are given some $T_{\mu\nu}$ in a spacetime with metric $g_{\mu\nu}$, you check whether is is a perfect fluid or not by finding a solution u^μ to the following equation

$$\left(\Lambda^{-1}\right)^\mu_{\ \rho} T \Lambda(u) = \tag{2.35}$$

[9]cfu: *What is $g_{\mu\nu;\gamma}$? Why? It vanishes in the LIF, therefore it vanishes everywhere*

[10]Remember that $X_{;\mu} \equiv \nabla_\mu X$ for any tensor X.

where $\Lambda = \Lambda(u)$ is a Lorentz transformation with velocity u^μ and $\mathbf{T} = T_{\mu\nu}$ If $T_{\mu\nu}$ is that of a perfect fluid, then you can derive ρ, p and u^μ from (see Prob. P.2.2)

$$\rho \equiv \frac{1}{4}\left(\sqrt{12T_{\mu\nu}T^{\mu\nu} - 3T^2} - T\right), \tag{2.36}$$

$$p \equiv \frac{1}{12}\left(\sqrt{12T_{\mu\nu}T^{\mu\nu} - 3T^2} + 3T\right), \tag{2.37}$$

$$u_\mu u_\nu \equiv \frac{T_{\mu\nu} - g_{\mu\nu}p}{\rho + p}, \tag{2.38}$$

where $T \equiv T^{\mu\nu}g_{\mu\nu} = T^\mu_\mu$.

The energy-momentum tensor is again covariantly conserved $T^{\mu\nu}_{;\nu} = 0$. This can be seen as the conserved Noether current for the diffeomorphism invariance of the matter action (see e.g. 19.6 [6]). Currents of gauge transformations (diffeomorphism are gauged by gravity) are identically conserved, and, in fact, $T^{\mu\nu}_{;\nu} = 0$ follows directly from Einstein Equations (2.27). In general, an equation of state $p = p(\rho)$ is necessary to close the system of equations. The extension to imperfect fluids is nicely discussed in B.10 of [51] and [47].

For the discussion of the Cosmic Microwave Background (CMB) and Large Scale Structures, we will have to consider more general "imperfect" fluids, with additional contributions to $T_{\mu\nu}$, that are organised in a derivative expansion as

$$T^{\text{imp.}} \sim T^{\text{perfect}}_{\mu\nu} + \sum_n d^n \nabla^n u \tag{2.39}$$

with some length scale d. The theory of general fluids, namely *hydrodynamics*, should then be thought of as a large scale effective theory, defined as an expansion in d/L, where L is the typical size of spatial variations in the fluid and in the classical examples d is the mean free path of the microscopic constituents.

If the fluid carries some *conserved charge* N, such as for example the number of particles, in the rest fram of the fluid one expects a charge density $n \equiv N/V$ for some small volume V that is conserved $\dot{n} \doteq 0$. To describe the conservation of n in any other frame, we must then have the covariant expression

$$(u^\mu n)_{;\mu} = 0, \tag{2.40}$$

which indeed reduced to $\dot{n} \doteq 0$ for $u^\mu = \{1, 0, 0, 0\}$. Since most processes in the universe are approximately adiabatic the total entropy of the universe is approximately conserved. An important covariantly conserved quantity is therefore the *entropy density* $s = S/V$, with entropy current su^μ. In general one finds

$$s = \frac{\rho + p - \mu n}{T}, \tag{2.41}$$

with n the number density of particles.

Relativistic kinetic theory See Lesson 6.

Problems for lesson 1

P.2.1 Compute the Christoffel symbols for a flat FLRW spacetimes.

$$ds^2 = -dt^2 + a^2 dx^i \delta_{ij} dx^j \tag{2.42}$$

P.2.2 Derive the covariant expressions for ρ, p (as in (2.36)) and u^μ in terms of $T^{\mu\nu}$ and $g_{\mu\nu}$. First build the two non-trivial scalars using $T^{\mu\nu}$ and $g_{\mu\nu}$. Then compute their value using the definition of $T_{\mu\nu}$ for a perfect fluid, (2.34), and invert to find p and ρ. Finally derive $u^\mu u^\nu$.

Check for understanding of Lesson 1

cfu.2.1 In special relativity the Lorentz invariant distance between any two point is $d^2 = \Delta x^\mu \eta_{\mu\nu} \Delta x^\nu$. What is the generalisation of this distance to GR? What distance does $g_{\mu\nu}(x)$ measure?

cfu.2.2 Study the geodesic equations, following appendix A of these lecture notes

$$\frac{d^2 x^\mu}{du^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{du} \frac{dx^\beta}{du} = 0\,. \tag{2.43}$$

cfu.2.3 For what phenomena in physics do I need General Relativity (GR), as opposed to Euclidean spacetime and Newtonian dynamics?

LESSON 3

A Homogeneous and Isotropic Expanding Universe

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In this lesson, I discuss FLRW spacetimes, fluids, cosmological equations of state, Friedmann, continuity and acceleration equations and the expansion of the universe.

3.1 Symmetric spaces

Most solutions of GR that we work with contain some amount of at least approximate symmetry. A metric enjoys an *isometry* if there is a change of variable $\tilde{x} = \tilde{x}(x)$ that leaves the metric unchanged in the sense

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x) \stackrel{!}{=} g_{\mu\nu}(\tilde{x})\,. \tag{3.1}$$

This is equivalent to (prove it)

$$ds^2(x) = g_{\mu\nu}(x) dx^{mu} dx^\nu \stackrel{!}{=} g_{\mu\nu}(\tilde{x}) d\tilde{x}^{mu} d\tilde{x}^\nu = ds^2(\tilde{x}(x))\,, \tag{3.2}$$

where by $ds^2(\tilde{x}(x))$ I mean that I substitute every x with an $\tilde{x}(x)$. Isometries are best discussed using Killing vectors. Given the change of coordinates $x'^\mu = x^\mu + \xi^\mu$, every tensor changes by minus its Lie derivative $\mathcal{L}_\xi$ (see Box 1) in the ξ direction. For the metric to be invariant we require[11]

$$\begin{aligned} \Delta g_{\mu\nu}(x) &\equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -\mathcal{L}_\xi g_{\mu\nu}(x) && (3.3)\\ &= -\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \stackrel{!}{=} 0\,. && (3.4) \end{aligned}$$

Vectors ξ for which $\mathcal{L}_\xi g_{\mu\nu} = 0$ leave the metric invariant and are called Killing vector fields, or simply *Killing vectors*. Remarkably, Killing vectors are completely determined by their value and that of their derivative at one point. To see this, recall two defining properties of the Riemann tensor

$$[\nabla_\mu, \nabla_\nu] V_\rho = R_{\rho\sigma\mu\nu} V^\sigma\,, \quad R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} + R_{\rho\mu\nu\sigma} = 0\,. \tag{3.5}$$

We can sum over cyclic permutations of the first equation, use the second equation as well as the definition of Killing Vectors (3.4) and find

$$\nabla_\rho \nabla_\sigma \xi_\mu = [\nabla_\rho, \nabla_\mu] \xi_\sigma = R_{\lambda\sigma\mu\rho} \xi^\lambda\,. \tag{3.6}$$

The solution of this second order pde are determined by the initial condition $\{\xi^\mu(\bar{x}), \nabla_\nu \xi^\mu(\bar{x})\}$ at some point $\bar{x}$ (at least locally one can construct the solution as a Taylor expansion in $x - \bar{x}$), and so takes the form

$$\xi_\mu(x) = A^\rho_\mu(x, \bar{x}) \xi_\rho(\bar{x}) + B^{\sigma\rho}_\mu(x, \bar{x}) \nabla_\sigma \xi_\rho(\bar{x})\,. \tag{3.7}$$

Since we can specify at most D independent $\{\xi^\mu(\bar{x})\}$ and $D(D-1)/2$ independent $\{\nabla_\nu \xi^\mu(\bar{x})\}$ (antisymmetry of $\nabla_\mu \xi_\nu$ follows from the Killing condition), the maximum number of a isometries a spacetime

[11] CFU: *Derive this expression taking advantage of the fact that the metric is covariantly constant, $\nabla_\mu g_{\nu\rho} = 0$, and using Eq. (??).*

can enjoy is $D(D+1)/2$, which reduced to 10 in $D=4$. Spaces that saturate this upper bound on the number of independent Killing vectors (isometries) are referred to as *maximally symmetric spaces*[12].

To gain some intuition on these $D(D+1)/2$ generators, let us choose the following $D(D+1)/2$ linearly independent initial conditions

$$\begin{cases} \xi^{(\alpha)}_\rho = \delta^\alpha_\rho\,, & \text{(D solutions)} \\ \nabla_\rho \xi^{(\alpha)}_\sigma = 0\,. & \\ \xi^{(\alpha\beta)}_\rho = 0\,, & \\ \nabla_\rho \xi^{(\alpha\beta)}_\sigma = \delta^\alpha_{(\rho}\delta^\beta_{\sigma)}\,. & (D(D-1)/2 \text{ solutions})\,, \end{cases} \tag{3.8}$$

where the indices α and β label the solutions. The first D solutions cover completely the tangent space of the manifold at point $\bar{x}$ and can hence be thought of as generalised[13] translations: they move the (arbitrary) point $\bar{x}$ in any of the D direction. It can be proven (see Ch 13 of [47]) that the remaining $D(D-1)/2$ Killing vectors change any vector $V^\mu(\bar{x})$ into any other vector $\tilde{V}^\mu(\bar{x})$ with the same norm $V^\mu V_\mu = \tilde{V}^\mu \tilde{V}_\mu$. These isometries can then be thought of as generalised[14] rotation. We conclude that a maximally symmetric space is *homogeneous* (invariant under generalised translations) and *isotropic* (invariant under generalised rotations)[15]. The converse is also true, i.e. all homogeneous and isotropic spacetimes are maximally symmetric as follows from a simple counting of isometry generators[16].

There are three more theorems that I have to quote without a detail proof :

- *Uniqueness:* Maximally symmetric spaces are uniquely characterised by the value of the Ricci scalar R, which is just a constant number over the space by homogeneity, and the signature of the metric (see 13.2 of [47]).
- For Maximally symmetric spaces the Riemann tensor is proportional to the metric

$$R_{\mu\nu\rho\sigma} = K g_{\mu(\sigma} g_{\nu\rho)}\,, \tag{3.9}$$

where K is related to the Ricci scalar by

$$R = -D(D-1)\,K\,. \tag{3.10}$$

- If a space M contains a maximally symmetric subspace $N \subset M$, the metric can always be written as the following "warp product"

$$ds^2 = g_{ab}(x)dx^a dx^b + f(x)\tilde{g}_{ij}(y)dy^i dy^j\,, \tag{3.11}$$

with $\tilde{g}_{ij}$ the metric of the maximally symmetric subspace, y the coordinates of the subspace and x the remaining coordinates.

Box 3.1 Lie derivatives This discussion is based on App. B of [?] and Ch. 8 of [6]. Consider a vector field V^μ on some manifold M, which we parameterise locally with coordinates x. The vector generates *integral curves*, i.e. solutions of

$$\frac{\partial x^\mu}{\partial t} = V^\mu(x)\,. \tag{3.12}$$

These curves are tangent to V^μ at every point. We can think of an integral curve $x^\mu(t)$ as a one parameter family of (finite) changes of coordinates $x^\mu(t_0) \to x'^\mu = x^\mu(t)$. Instead of a passive coordinate transformation in which tensors on the manifold remain fixed, and points of the manifold change name according to the change of coordinates above, we can define an active transformations in which we drop the prime from the

[12] cfu: *If I can freely specify the initial conditions $\{\xi^\mu(\bar{x}), \nabla_\nu\xi^\mu(\bar{x})\}$, why are not all spaces maximally symmetric? There are "integrability" restrictions, which depend on the metric, on the set of initial data $\{\xi^\mu(\bar{x}), \nabla_\nu\xi^\mu(\bar{x})\}$ that admits a solution (see 13.1.12 of [47]).*

[13] "Generalised" here means both that these act as translations only locally, rather than globally, and that it is not specified whether the coordinate they translate are Euclidean or not.

[14] "Generalised" here also refers to the fact that when the signature of the metric is e.g. Lorentzian, rather than Minkowskian, these isometries correspond locally to boosts rather than rotations.

[15] cfu: *Does homogeneity imply isotropy? (no, e.g. this room with constant gravitational field) Does isotropy imply homogeneity? (no around a single point) Does isotropy around every point imply homogeneity? (yes)*

[16] CFU: *Prove this. Prove also that the number of Killing vectors does not depend on the choice of coordinates.*

new coordinates x' and impose a change of all the tensors with fixed coordinates, i.e. a diffeomorphism

$$x \to x' \,, \quad T(x) \to T(x') \quad \text{(passive change of coords)}\,, \tag{3.13}$$
$$T(x) \to T'(x') \equiv T(x(x')) \quad \text{(active diffeomorphism)}\,. \tag{3.14}$$

Then, we can ask how a given tensor changes under infinitesimal diffeomorphism generated by an integral curve. We define the Lie derivative $\mathcal{L}$ of any covariant tensor $T^{\cdots}_{\cdots}$ (i.e. transforming as in Eq. (2.4)), in the V^μ direction is given by

$$\mathcal{L}_V T^{\cdots}_{\cdots}(x) \equiv \lim_{\epsilon\to 0} \frac{T^{\cdots}_{\cdots}(x) - T'^{\cdots}_{\cdots}(x)}{\epsilon}\,, \quad \text{with} \quad x'^\mu(x) = x^\mu + \epsilon V^\mu(x)\,. \tag{3.15}$$

As the name suggests, this derivative is a linear operator and obeys the Leibniz rule

$$\mathcal{L}_V(aT + bS) = a\mathcal{L}_V T + b\mathcal{L}_V S\,, \quad \mathcal{L}_V\,(T * S) = (\mathcal{L}_V T) * S + T * (\mathcal{L}_V S)\;, \tag{3.16}$$

where $*$ represents any index contraction. For scalar, vector and tensor field one finds

$$\begin{aligned}
\mathcal{L}_V\phi &= V^\mu\partial_\mu\phi\,, && (3.17)\\
\mathcal{L}_V W^\mu &= V^\nu\partial_\nu W^\mu - W^\nu\partial_\nu V^\mu = V^\nu\nabla_\nu W^\mu - W^\nu\nabla_\nu V^\mu\,, && (3.18)\\
\mathcal{L}_V W_\mu &= V^\nu\partial_\nu W_\mu + W_\nu\partial_\mu V^\nu = V^\nu\nabla_\nu W^\mu + W_\nu\nabla_\mu V^\nu\,, && (3.19)\\
\mathcal{L}_V T_{\mu\nu} &= V^\rho\nabla_\rho T_{\mu\nu} + T_{\rho\nu}\nabla_\mu V^\rho + T_{\mu\rho}\nabla_\nu V^\rho\,. && (3.20)
\end{aligned}$$

where the intermediate expressions makes it explicit that the Lie derivatives are independent of the metric. Notice that the Lie Derivative is still a tensor of the same rank as suggested by Eq. (3.15). For vectors $\mathcal{L}_V W = -\mathcal{L}_W V$. In particular, since the metric is symmetric and covariantly constant, one finds

$$\mathcal{L}_V g_{\mu\nu} = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu\,. \tag{3.21}$$

Since isometries are defined by $g'(x) = g(x)$, this give the Killing equation for the generators of isometries

$$\nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu = 0\,. \tag{3.22}$$

Note that any linear combination of Killing vectors is itself a Killing vector and so generates isometries.

3.2 The Friedmann-Lemaitre-Robertson-Walker metric

The Cosmic Microwave Background (CMB) radiation (representing the universe 370,000 years after the Big Bang) appears isotropic to a part in 10^5 (see e.g. Fig. 2). The distribution of galaxies on scales much larger than 5 Mpc is homogeneous (see Fig. 3) (inhomogeneities go from 10^{-5} on Hubble scales to $\mathcal{O}(1)$ at around 5 Mpc). Both type of observations indicate that there exist a choice of coordinates in our universe such that constant time hypersurfaces are approximately homogeneous and isotropic. Using the theorem reported in the previous section, the metric of our universe therefore must be approximated on large scales by

$$ds^2 = -dt^2 + a(t)^2\tilde{g}_{ij}(x,K)dx^i dx^j\,. \tag{3.23}$$

where $\tilde{g}_{ij}(x,K)$ is the metric of the maximally symmetric spatial 3D hypersurface, which we derive in the following. A few comments are in order

- I can always re-define time as above to ensure that $g_{00}(t) = -1$. This choice is called *cosmological time* (or often simply time) and, as we will see in the next lecture, corresponds to the proper time of observers at rest in the coordinates x^i.
- x are *comoving coordinates* as opposed to physical ones. A spacelike comoving interval Δx^i is related to physical distance by a factor of a:

$$\Delta x_{phys} = \sqrt{\Delta x^\mu g_{\mu\nu}\Delta x^\nu} = \sqrt{a^2\Delta x^i\tilde{g}_{ij}\Delta x^j} = a|\Delta x|\,. \tag{3.24}$$

- The simplest possibility would of course be a constant $a(t)$, which could then be re-absorbed into the definition of x. But this is not what we observe. Since the early 20th century we know that in average all nearby ($\ll$ Gpc) galaxies recede from us at a speed proportional to their distance. This was originally pointed out by the influential plot by Edwin Hubble reported in Fig. 1 and leads to the mathematical relation called the *Hubble law*

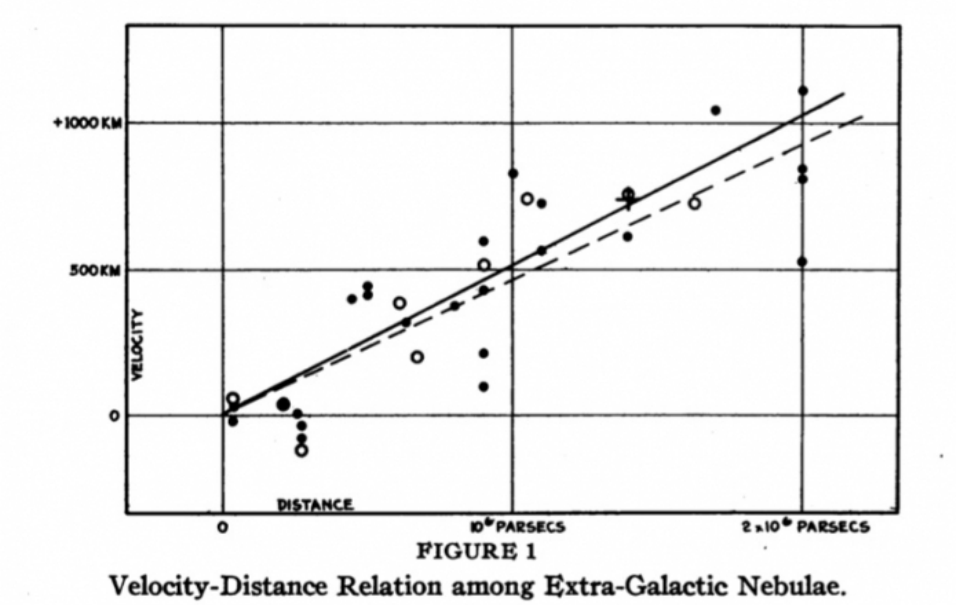


Figure 1: Original plot of Hubble's data on the distance (horizontal axis) and velocity (vertical axis) of nearby galaxies.

$$v \equiv \dot{x}_{phys} = H_0 x \,, \tag{3.25}$$

with some constant H_0 called the Hubble constant. Using (3.24) the Hubble law gives

$$\dot{x}_{phys} = \partial_t \left[a(t)|x|\right] = \dot{a}|x| = \frac{\dot{a}}{a}a|x| = \frac{\dot{a}}{a}x_{phys} \stackrel{!}{=} H_0 x_{phys} \tag{3.26}$$

where all time dependent quantities are evaluated today, $t = t_0$. We conclude that

$$\dot{a}(t_0)/a(t_0) = H_0 > 0 \,, \tag{3.27}$$

and so the universe is expanding.

We want now to determine the explicit form of the 3D spatial metric $\tilde{g}_{ij}$. It is actually sufficient to find *one* maximally symmetric spatial (i.e. with all plus signature) metric with curvature K. Because of the uniqueness theorem in the last section, all other possible maximally symmetric spatial metrics with curvature K are related to this one by a change of coordinates (i.e. the spacetime is the same). A very simple procedure is then to consider Euclidean space in one more dimension, i.e. $D = 4 + 0$, and derived the induced metric on the well-known constant curvature objects: the sphere ($K > 0$), the plane ($K = 0$) and the hyperboloid ($K < 0$). To minimise the use of indices and maximise transparency, I'll do this for a 2D surface embedded in 3 spatial dimensions. A generalization to any number of dimensions is straightforward and is left as an exercise.

Let us start with a 2-sphere in flat space

$$R^2 = x^2 + y^2 + z^2 \,, \quad ds^2 = dx^2 + dy^2 + dz^2 \,. \tag{3.28}$$

The induced metric is simply derived from the embedding

$$dz = -\frac{xdx + ydy}{\sqrt{R^2 - x^2 - y^2}} \,. \tag{3.29}$$

Going to "polar" coordinates, one gets

$$\left\{ \begin{array}{l} x = \tilde{r}\cos\phi \,, \\ y = \tilde{r}\sin\phi \end{array} \right. \quad \Rightarrow \quad \begin{array}{rl} dl^2 = & \frac{d\tilde{r}^2}{1-\tilde{r}^2/R^2} + \tilde{r}^2 d\phi^2 \\ = & R^2\left[\frac{dr^2}{1-Kr^2} + r^2 d\phi^2\right] \,. \end{array} \tag{3.30}$$

Generalizing to our universe [Problem P.4.2] one finds

$$ds^2 = -dt^2 + a^2\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega_2^2\right] \tag{3.31}$$

$$= -dt^2 + a^2\left[d\chi^2 + f(\chi)d\Omega_2^2\right] , \tag{3.32}$$

$$f(\chi) = \begin{cases} \sinh(\chi)^2 & K = -1 \,(\text{open hyperbolic space}) , \\ \chi^2 & K = 0 \,(\text{flat space}) , \\ \sin(\chi)^2 & K = +1 \,(\text{closed space or sphere}) . \end{cases} . \tag{3.33}$$

where K is the *spatial curvature* and

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2 . \tag{3.34}$$

Notice that $\chi \in \{0, \infty\}$ for open and flat universe, while $\chi \in \{\pi, 0\}$ for closed universes[17][18]. Also for flat universe $K = 0$ there is an ambiguity due to the rescaling $\{r, a\} \to \{\lambda r, \lambda^{-1} a\}$, which leaves the metric invariant. This is often fixed by imposing the additional condition $a_0 = 1$ (for $K \neq 0$ this rescaling is fixed by normalizing $K = \pm 1$). It is sometimes convenient to have the metric in quasi-Cartesian coordinates as well, as opposed to spherical ones, namely[19]

$$ds^2 = -dt^2 + a(t)^2 \frac{dx^i dx^j \delta_{ij}}{(1 + K\mathbf{x}^2/4)^2} \tag{3.35}$$

$$= -dt^2 + a(t)^2 d\tilde{x}^i d\tilde{x}^j \left[\delta_{ij} + K\frac{\tilde{x}_i \tilde{x}_j}{1 - K\tilde{\mathbf{x}}^2}\right] . \tag{3.36}$$

There is no evidence of spatial curvature in our universe and current upper bounds constrain it to be at a sub-percent level (see (P.9.4) for a precise statement). For this reason, in these introductory notes I will mostly focus on the flat case, $K = 0$. For future reference let us report the flat FLRW metric [homework P.3.3]

$$ds^2 = -dt^2 + a^2(t) dx^i \delta_{ij} dx^j , \tag{3.37}$$

$$= a^2(t)\left[-d\tau^2 + dx^i \delta_{ij} dx^j\right] , \tag{3.38}$$

where in the second line I introduces to *conformal time*, $ad\tau \equiv dt$, which makes manifest that flat FLRW is conformally equivalent to Minkowski.

It is important to appreciate that FLRW as $3+1$D manifold is *not* maximally symmetric, since time translations and boost are broken by the time dependent of the scale factor $a(t)$. It is the constant-time 3D hypersurface that is maximally symmetric.

For a flat FLRW metric, many Christawful symbols vanish by the isotropy (rotational invariance) of the FLRW spacetime[20]. The other Christoffel symbols are [homework P.2.1]

$$\Gamma^0_{ij} = Ha^2\delta_{ij} , \quad \Gamma^j_{i0} = \Gamma^i_{j0} = H\delta_{ij} , \quad \Gamma^0_{00} = 0 . \tag{3.39}$$

3.3 Dynamical equations

Continuity equation Let us focus on homogeneous and isotropic fluids, as relevant to describe an FLRW background[21]. The most general homogeneous and isotropic two tensor takes the form

$$T^\mu{}_\nu = \text{Diag}\left\{-\rho, p, p, p\right\} , \quad T_{\mu\nu} = \text{Diag}\left\{\rho, a^2 p, a^2 p, a^2 p\right\} , \tag{3.40}$$

where we will interpret ρ as the energy density (units of E/L^3) and p as the pressure (units $M/(T^2L) = E/L^3$).

[17] cfu: *What is the sum of the internal angles in a triangle?*

[18] cfu: *What is the volume of a spatial slice? It is finite only for a sphere*

[19] CFU: *Derive the relation between x^i and $\{r, \theta, \phi\}$.*

[20] cfu: *What Christoffel symbols of FLRW vanish by symmetry? First, verify that Γ transforms as a tensor for affine changes of coordinates $x'^{\mu'} = M^{\mu'}_\mu x^\mu + C^{\mu'}$ with C and M constant. This includes global rotations (both boosts and rotations) and spacetime translations. Only spatial rotations and spatial translations are symmetries of FLRW. The only tensor invariant under rotations is δ_{ij}. So $\Gamma^i_{00} = \Gamma^0_{0i} = 0$. Translations imply that $\Gamma = \Gamma(t)$. $\Gamma^0_{00} = 0$ is an "accident" of choosing the proper time of comoving observes t, instead of say τ or other $t' = f(t)$.*

[21] cfu: *Reflect on the fact that the simple diagonal form above arises in two unrelated contexts. It follows from homogeneity and isotropy for any energy-momentum tensor, not just that of a perfect fluid. It arises in the locally inertial Cartesian frame of perfect fluids, for arbitrarily anisotropic and inhomogeneous configurations.*

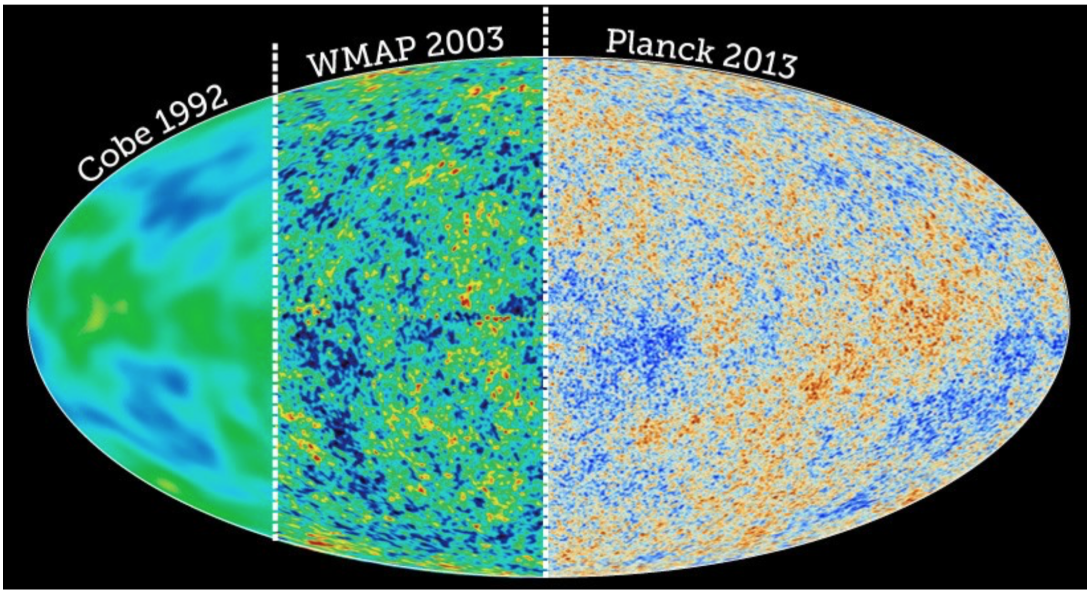


Figure 2: The temperature anisotropies in the CMB as seen by COBE ($l_{max} \sim 20$, $\theta_{min} \sim 9°$), WMAP ($l_{max} \sim 800$, $\theta_{min} \sim 0.2° \simeq 12\,$arcmin) and Planck ($l_{max} \sim 2500$, $\theta_{min} \sim 0.07 \simeq 4\,$arcmin).

EE's imply the covariant conservation of energy and momentum current, i.e. $T^{\mu\nu}_{;\nu} = 0$. The spatial components of this equation are trivial because of isotropy. Instead the time component plays a crucial role in cosmology:

$$\boxed{\dot\rho + 3H\,(\rho + p) = 0}\,. \tag{3.41}$$

The one-parameter family of *equation of state*

$$p = w\rho\,, \tag{3.42}$$

with constant w is used constantly in cosmology. For these linear equations of state, it is easy to solve the continuity equation implicitly

$$\dot\rho + 3\frac{\dot a}{a}(1+w)\rho = 0 \quad \Rightarrow \quad \rho(t) = \rho(t_0)\left[\frac{a(t)}{a(t_0)}\right]^{-3(1+w)}\,. \tag{3.43}$$

For example:

- For non-relativistic matter, or *dust*, $p = aP \simeq mv \ll mc \simeq E$ and therefore $p \ll \rho$ or $w \ll 1$. Expansion leads to $\rho \propto a^{-3}$
- For relativistic matter, or *radiation*, we have $P \simeq E$ and therefore $p = \rho/3$ or $w = 1/3$ (see around (5.1) for a detailed derivation). Expansion leads to $\rho \propto a^{-4}$
- For a cosmological constant, or *vacuum energy*, $T_{\mu\nu} = \Lambda g_{\mu\nu}$ and therefore $p = -\rho$ or $w = -1$. Expansion leads to $\rho \propto a^0 \sim const$

A simple interpretation of these scaling is that of an expanding box of linear size $a(t)$. Non-relativistic matter density dilutes with the volume, i.e. a^{-3}. Relativistic matter, aka radiation also dilutes with the volume as a^{-3}, but it has an extra a^{-1} suppression due to the redshift of the momentum of each particle (and the mass is negligible). Vacuum energy does not dilute[22].

Friedmann equation Let solve the EE's for an FLRW metric. Using the definition of the Riemann and Ricci tensors in (2.17), and the FLRW metric (3.35), a lengthy but straightforward computation in (quasi-)Cartesian coordinates shows

$$R_{00} = 3\frac{\ddot a}{a}\,, \quad R_{ij} = -\delta_{ij}\frac{2K + 2\dot a^2 + a\ddot a}{\left(1 + K\mathbf{x}^2/4\right)^2}\,, \quad R = -6\left[\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{K}{a^2}\right]\,, \tag{3.44}$$

[22]cfu: *What is the time dependence of $\rho(a)$ for infinitely long cosmic strings? And for a domain wall?*

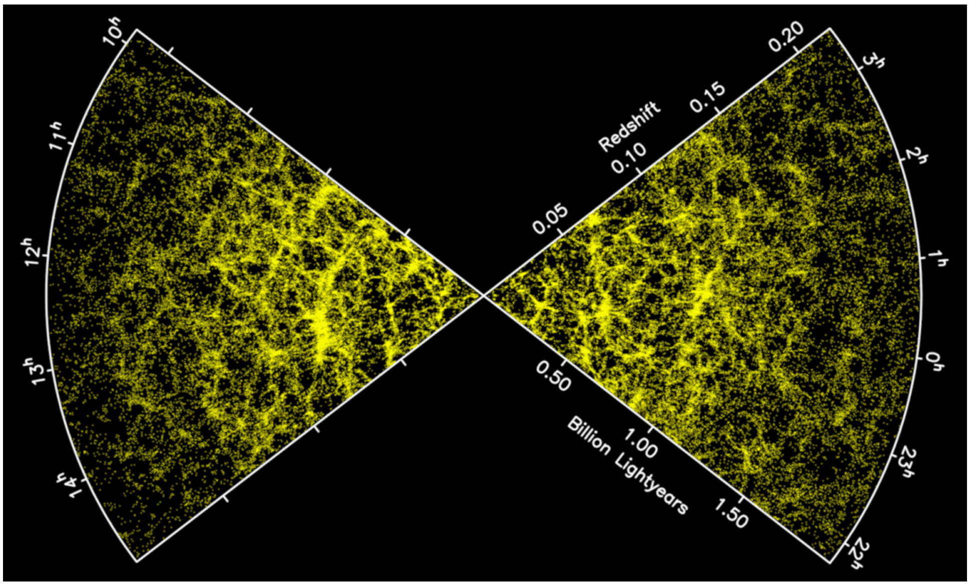


Figure 3: The distribution of galaxies as measured by the Sloan digital Sky Survey. The 3D dimension animated version is available here. In assessing homogeneity, keep in mind that more distant galaxies correspond to earlier time, when structures had had less time to grow.

The 00-component of the EE's in (2.27) is the easily derived

$$\boxed{3M_{\rm Pl}^2\left(H^2+\frac{K}{a^2}\right)=\sum_i \rho_i\,,} \tag{3.45}$$

where i runs over all constituents of the universe (radiation, DM, neutrinos and baryons). This is often called the *Friedmann equation*. Notice that since a (flat) FLRW metric is specified by a single function $a(t)$, we need only one of the ten EE's to determine the solution. It is then convenient to divide everything by the *critical density* (a function of time)

$$\rho_c \equiv 3M_{\rm Pl}^2 H^2\,, \tag{3.46}$$

and find

$$1-\Omega_k=\sum_a \Omega_a\,, \quad \text{with} \quad \Omega_k\equiv -\frac{K}{H^2a^2}\,, \quad \Omega_a\equiv\frac{\rho_a}{\rho_c}\,. \tag{3.47}$$

Notice that only Ω_k can be negative.

Using a to parameterize time and solving for $H(a)$ gives[23] (see P.3.4)

$$H=\frac{\dot a}{a}=\sqrt{\frac{\rho}{3M_{\rm Pl}^2}}=H_0\left(\frac{a_0}{a}\right)^{3(1+w)/2}\Rightarrow a(t)=\left[\frac{3}{2}\left(1+w\right)H_0 t\right]^{\frac{2}{3(1+w)}} \tag{3.48}$$

for $w\neq -1$, where one fixes the integration constant requiring that a vanishes at past infinity.

Important solutions for the scale factor are then

- For non-relativistic matter, or *dust*, $w\simeq 0$ so $a\propto t^{2/3}$
- For relativistic matter, or *radiation*, $w=1/3$ so $a\propto t^{1/2}$

[23] cfu: *Is this a fully non-linear exact solution of EE's? Yes.*

- For a cosmological constant, or *vacuum energy*, $w = -1$ this expressions is singular. One finds $a \propto e^{H_0 t}$ (see P.3.4)

Notice that if a is a monomial in t one finds always $H \propto t^{-1}$, or more precisely

Box 3.2 Null Energy Condition (NEC) A certain form of matter with energy-momentum tensor $T_{\mu\nu}$ satisfies the Null Energy Condition if for ever null vector $N^\mu N_\mu = 0$ one has

$$T_{\mu\nu} N^\mu N^\nu \geq 0 \quad (\text{NEC})\,. \tag{3.49}$$

Using the perfect fluid parameterization in (2.34), this implies $\rho + p \geq 0$. Violations of the NEC are often associated with pathologies such as ghosts instabilities (i.e. field with the wrong-sing kinetic term that can be nucleated by decreasing the energy of the system) or tachyon instabilities [15]. Yet, more exotic theories with non-standard kinetic terms, such as the ghost condensate, are known to safely violate the NEC, see e.g. [10, 42].

$$H(t) = \frac{2}{3(1+w)} \frac{1}{t}\,. \tag{3.50}$$

This gives the *age of the universe* for this simple universe (valid for any single-fluid cosmology, see 4.25 for a general derivation)

$$t_{age} = \frac{2}{3(1+w)} \frac{1}{H(t_{age})}\,, \tag{3.51}$$

There are two other combinations of EE's that come in handy. First, subtracting the 00 EE from the (summed) ii EE's, one finds the *acceleration equation*

$$M_{\rm Pl}^2 \frac{\ddot{a}}{a} = -\frac{1}{6} (\rho + 3p)\ . \tag{3.52}$$

Second, by taking the time derivative of the Friedmann equation and using the continuity equation to get rid of $\dot{\rho}$, we can find the variation of the Hubble parameter

$$-\dot{H} M_{\rm Pl}^2 = \frac{1}{2} (\rho + p)\ . \tag{3.53}$$

Most cosmological "stuff" obeys the Null Energy Condition (3.49), and so H decreases during cosmic evolution.

Charge conservation How does a conserved charge density depend on time in an FLRW universe? Isotropy implies that the normalised velocity defined in (2.32) and associated to the associated conserved current must take the form $u^\mu = \{1, \vec{0}\}$. Then, the conservation of the current $(n u^\mu)_{;\mu} = 0$ reduces simply to

$$\dot{n} + 3Hn = 0\,, \tag{3.54}$$

which admits the solution (see P.3.1)

$$n(t) = \left[\frac{a(t_0)}{a(t)}\right]^3 n(t_0) \propto \frac{1}{a^3}\,. \tag{3.55}$$

Problems for lesson 2

P.3.1 Compute the evolution of the entropy density $s(a)$ in an FLWR universe in the (good) approximation that all processes are adiabatic. How does this compare to the evolution of any other conserved charge?

P.3.2 Compute and solve the geodesic equation for a massless particle in FLRW

P.3.3 Using the definition of isometry for a coordinate transformation $\tilde{x}(x)$, namely

$$\tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(\tilde{x})\,, \tag{3.56}$$

verify that the (flat) FLRW metric

$$ds^2 = -dt^2 + a^2 dx^i \delta_{ij} dx^j \tag{3.57}$$

is indeed homogeneous and isotropic (i.e. isometric with respect to spatial translations $\tilde{x}^i = x^i + b^i$ and rotations $\tilde{x}^i = R^i_j x^j$).

P.3.4 Solve the Friedmann equation for for $w = -1$ and $w \neq -1$.

P.3.5 Consider an FLRW universe with a single fluid and derive the acceleration equation Eq. (3.52) for $\ddot{a}(t)$. You can either derive the ii component of the Einstein equations or use the Friedman equation together with the covariant conservation of energy, $T^{0\mu}_{;\mu} = 0$. For what w does one get accelerated expansion?

Check for understanding of Lesson 2

cfu.3.1 Study the following equations, following appendix A of the lecture notes

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + g^{\mu\nu} p\,, \tag{3.58}$$

$$\frac{d^2 x^\mu}{du^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{du} \frac{dx^\beta}{du} = 0\,. \tag{3.59}$$

cfu.3.2 What is a fluid? At what distance can I describe air in this room as a fluid?

cfu.3.3 After having solved the geodesic equation for a massless particle in problem P.3.2, discuss how conservation of energy works out. In particular, where does the energy of photon in an expanding (or contracting) FLRW universe go?

cfu.3.4 How does the continuity equation Eq. (3.41), which expresses the covariant conservation of energy compare with the covariant conservation of a charge density? How can I visualize this in terms of things flowing in or out of a fixed region?

LESSON 4

Redshift and distances

ref

Important distances: particle horizon, luminosity (and magnitude) and angular diameter distances, age of the universe. Constituents of the universe: curvature, photons, baryons, neutrinos, dark matter and dark energy.

Cosmological redshift Olbers' paradox is the argument that the universe cannot be eternal and infinite because otherwise the night sky should be bright, since every direction in the sky would point to some star with a similar intrinsic luminosity as the sum. In the Big Bang model, the universe has a finite age (about 13.7 billion years). This actually makes the problem worse since the universe must have been much hotter in the past and we should see an even brighter sky, say from very hot thermal radiation. The resolution is that in FLRW the wavelength of light redshifts[24] $E \sim \lambda^{-1} \sim a^{-1}$. This can be seen directly from the geodesic deviation discussed in 1, but I'll give a different derivation.

Recall that redshift is defined as

$$1 + z \equiv \frac{\lambda_o}{\lambda_e}\,, \tag{4.1}$$

where e and o stand for emission and observation, respectively. Consider now the light propagating to us along the $-\hat{r}$ direction from some emitting source at comoving position $\{r, \theta, \phi\}$ in spherical coordinates. Photons are massless and so follow null geodesics with null tangent vector

$$ds^2 = 0 \quad \Rightarrow \quad \frac{dt}{a} = dr\,. \tag{4.2}$$

Consider a wave crest being emitted at some time t_e and arriving at time t_o to the observer at the origin $r = 0$. Then

$$\int_{t_e}^{t_o} \frac{dt}{a} = \int_0^r dr = r\,. \tag{4.3}$$

Consider now the subsequent wave crest being emitted at some time $t_e + \lambda_e$ (recall our units $c = 1$) and arriving at time $t_o + \lambda_o$ to the observer at the origin $r = 0$. We have similarly

$$\int_{t_e+\lambda_e}^{t_o+\lambda_o} \frac{dt}{a} = \int dr = r\,. \tag{4.4}$$

Subtract (4.3) from (4.4) and find

$$\int_{t_e}^{t_e+\lambda_e/c} \frac{dt}{a} = \int_{t_o}^{t_o+\lambda_o/c} \frac{dt}{a} \tag{4.5}$$

Performing the integral on each side under the approximation that a doesn't change much[25] we find

$$\frac{\lambda_e}{a_e} = \frac{\lambda_o}{a_0} \Rightarrow \frac{\lambda_o}{\lambda_e} = \boxed{1 + z = \frac{1}{a_e}}\,, \tag{4.6}$$

where I used the fact that in all practical application the observer is us today and so $a_0 = 1$ by convention. Notice that cosmological redshift is not Doppler redshift. The two agree *only at linear order*, i.e. $v \simeq zc + \mathcal{O}(z^2)$ (where I stressed the speed of light c) on distances much smaller than the spacetime curvature (H^{-1} for FLRW), because of dimensional analysis and the equivalence principle. To see this requires the definition of comoving distance Eq. (4.13) discussed below.

$$\begin{aligned} V &= H_0 a_0 \chi(z) = H_0 \int \frac{dt}{a} = H_0 \int \frac{dz}{H(z)} \simeq z + \mathcal{O}(z^2) \quad \text{(cosmoredshift)} && (4.7) \\ 1 + z &= \gamma \left(1 + \frac{V}{c}\right) \quad \to \quad v \simeq z + \mathcal{O}(z^2) \quad \text{(Dopplerinspecial relativity)}\,. && (4.8) \end{aligned}$$

[24] cfu: *Where does the energy of the gravitationally redshifted photon go?*
[25] CFU: *Estimate the change in $a(t)$ during one period of visible light*

Box 4.1 Geodesics in FLRW In FLRW, consider a massless particle

$$P^\mu = \left(E, P^i\right) = \partial_u x^\mu\,, \quad P^\mu P_\mu = 0 \quad \Rightarrow \quad E^2 = P^i g_{ij} P^j\,, \quad \text{and} \quad \partial_u = E\partial_t\,, \tag{4.9}$$

where u is the affine parameter of the photon geodesic and P^i is comoving momentum. Using the Christoffel symbols Γ^0_{ij} (see (3.39) and homework P.2.1), the 0-component geodesic equation is [homework P.3.2]

$$\frac{d^2x^0}{du^2} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{du}\frac{dx^\beta}{du} = E\left(\dot{E} + HE\right) = 0\,, \tag{4.10}$$

and therefore $E(t) = E(t_0)a(t_0)/a(t)$ and $P^i \propto a^{-2}$. Note that the physical momentum scales as the energy $\sqrt{P^i g_{ij} P^j} \propto a^{-1}$, as expected.

4.1 Distances

In non-relativistic mechanisc there are many different ways to measure distance, such as using a ruler, observing the apparent size of a object of known intrinsic size, observing the luminosity of a known "standard" candle and so on. In general relativity, all of measurements give different answers, so we have many different concepts of "distance" depending on how it is determined operationally. In cosmology there are several important distances that are used for example to probe the expansion history of the universe. All distances are conveniently related to comoving coordinates with appropriate factors of a. For an FLRW metric, the *comoving distance* is the distance travelled by a photon in a certain time interval in comoving coordinates. Consider spherical comoving coordinates

$$ds^2 \;\;=\;\; a^2\left[-d\tau + d\chi^2 + \chi^2\left(d\theta^2 + \sin\theta^2 d\phi^2\right)\right]\,, \tag{4.11}$$

and recall that for null geodesics $ds^2 = a^2\left[-d\tau^2 + d\eta^2\right] = 0$. Then $d\tau = d\eta$ and $\tau = \eta + \text{const}$. For the comoving distance one finds

$$\chi(t_i, t_f) = \int d\tau = \int_{t_i}^{t_f}\frac{dt}{a(t)} = \int\frac{da}{a^2H(a)} = \int\frac{dz}{H(z)}\,. \tag{4.12}$$

A common case is when the photon arrive on earth now and so $t_f = t_0$ or $a(t_f) = a_0 = 1$. Then one finds the comoving distance to a given redshift

$$\boxed{\chi(z) = \int_0^z\frac{dz}{H(z)}\,.} \tag{4.13}$$

The *luminosity distance* is useful as a measurement of the expansion of the universe when observing an object of known luminosity. The intrinsic luminosity L is the total amount of energy radiated per unit time. In Euclidean geometry, the intrinsic luminosity is related to the observed luminosity l by

$$l \equiv \frac{L}{4\pi d_L^2}\,, \tag{4.14}$$

where d_L is the luminosity distance and the factor of 4π comes about because l is defined as observed energy per unit time per unit surface. Unfortunately astronomers use a conventions dating back to the seventh century AD. Ptolemy made a survey of stars visible to the naked eye and divided them in six groups, from bright in group one to faint in group six. Later, in the 1800 Pogson made this mathematically precise introducing *magnitude*, which is related logarithmically to luminosity (lower magnitude is brighter)

$$m \equiv -2.5\log l + const\,, \tag{4.15}$$

and absolute magnitude is usually defined as observed magnitude at the distance of 10 pc. For a reference, $m_{sum} = -27$, $m_{sirius} = -1$, $m_{andromeda} = 0$ and the faintest thing[26] we can see by eye is $m < 6$.

To obtain Eq. (4.14) in an expanding universe we have to account for three factors:

- The comoving distance from emitting source and observation is $\chi(t_e, t_o)$ and it gets a factor of a_o to become a physical distance

[26] cfu: *What is the faintest magnitude we can see with telescopes, e.g. Hubble Space Telescope (HST)? The limit of HSC using visible light is* $m \leq 32$.

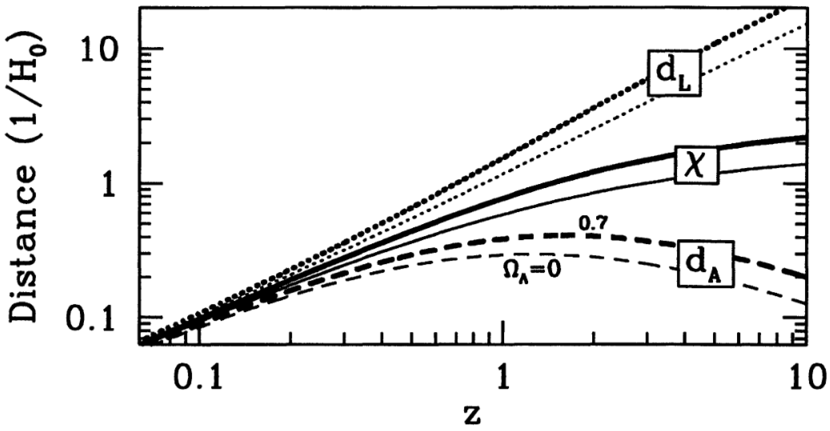


Figure 4: Angular diameter, comoving and luminosity distances as function of redshift. Thicker lines represent a flat universe with $\Omega_\Lambda = 0.7$, while thinner refer to $\Omega_\Lambda = 0$.

- the rate of arrival of photons (remember l has units of energy per *unit time* and unit area) decreases by a factor of a_e/a_o. Assuming $a_o = a_0 = 1$ this becomes $a_e/a_o = a_e = (1+z)^{-1}$
- the energy of incoming photons (remember l has units of *energy* per unit time and unit area) is redshifted by another factor of $a_e/a_o = (1+z)^{-1}$

Putting things together

$$l = \frac{L}{4\pi\left(\chi a_o\right)^2}\left(\frac{a_e}{a_o}\right)^2 \equiv \frac{L}{4\pi d_L^2} \quad \Rightarrow \boxed{d_L(z) = \chi a_o\left(\frac{a_o}{a_e}\right) = (1+z)\, a_o\chi(z)}\,, \tag{4.16}$$

where people usually set $a_o = a_0 = 1$ by redefining coordinates appropriately, and z is the redshift of emission.

The *angular diameter distance* in Euclidean geometry is defined for objects of size s subtending and angle θ from the point of view of the observer by $s \equiv d_A\theta$. Here there is only one factor of a in the relation of comoving to physical distance, hence

$$d_A(z) = \frac{\chi(z)}{1+z}\,. \tag{4.17}$$

The angular diameter distance is less useful in cosmology than the luminosity distance because object such as supernovae or galaxies at cosmological distances do not have well defined edges from which to extract a size. The various distances are summarized in Fig. 4.
[27]

Particle horizon Let us define the *particle horizon* $d_{\rm p.h.}$ the maximum (physical, as opposed to comoving) distance that light can have traveled since some "beginning of time" t_i (which could also be infinite). Any place further than that, at distance $d > d_{\rm p.h.}$ cannot have sent us any signal. We are not inside their future light cone, they are not in our past light cone. The particle horizon is then given by [Problem P.4.1]

$$d_{\rm p.h.}(t) \equiv a(t)\chi(t_i,t) = a(t)\tau(t_i,t) = a(t)\int_{t_i}^{t}\frac{dt'}{a(t')}\,, \tag{4.18}$$

where χ is the comoving distance and $a(t)$ transforms it into a physical distance. To gain intuition let us consider some simple single component universe, for which the scale factor is a power law in time (3.48),

[27] cfu: *Notice that the angular diameter distance in our universe starts decreasing around $z \sim 1$. Think of what happens to d_A for a fixed size segment on the surface of a sphere (or the earth), as you move it away from an observer at the north pole. What happens as you pass the equator?*

$a \propto t^{2/(3+3w)}$, with $w > -1$ for expansion (the case $w = -1$ is straightforward but requires a separate discussion) and beginning of time $t_i = 0$. Then

$$d_{\text{p.h.}}(t) = t^{2/(3+3w)} \int_0^t \frac{dt'}{t'^{2/(3+3w)}} = \frac{t^{2/(3+3w)}}{1 - 2/(3+3w)} \left[t^{1-2/(3+3w)} - 0^{1-2/(3+3w)}\right] . \tag{4.19}$$

For $2/(3+3w) > 1$, or equivalently $w < -1/3$ this diverges, while it converges to $d_{\text{p.h.}} \propto t$ (as expected by dimensional analysis) for $w > -1/3$. For example, $d_{\text{p.h.}} = 3t$ for matter ($w = 0$) and $d_{\text{p.h.}} = 2t$ for radiation ($w = 1/3$).

Age of the universe The *age of the universe* is computed from

$$t_{\text{age}} \int dt = \int \frac{da}{\dot{a}} = \int \frac{da}{aH(a)} = \int \frac{da}{a} \left[\frac{\sum_i \rho_i}{3M_{\text{Pl}}^2}\right]^{-1/2} , \tag{4.20}$$

where $H(a)$ is derived from the Friedmann equation. Notice that one does not need $a(t)$ here.

It is often convenient to use the dimensionless *fractional energy densities*[28] Ω_i instead of the dimensionful ρ_a. This is obtained by dividing $\rho_a(t)$ by the *critical energy density*

$$\rho_{crit}(t) \equiv 3M_{\text{Pl}}^2 H^2(t) \quad \Rightarrow \quad \Omega_a \equiv \frac{\rho_a(t)}{3H^2(t)M_{\text{Pl}}^2} . \tag{4.21}$$

The fractional energy densities at the present time ($a = a_0$), indicated by a subscript 0, are worth remembering:

$$\Omega_{\Lambda,0} \equiv \Omega_\Lambda(t_0) = 0.72 , \quad \Omega_{b,0} = 0.04 \quad \text{and} \quad \Omega_{DM,0} \simeq 0.24 . \tag{4.22}$$

The time evolution is then simply given by

$$\rho_i(t) = 3M_{\text{Pl}}^2 H_0^2 \frac{\Omega_{i,0}}{a(t)^{3(1+w)}} . \tag{4.23}$$

Nota that it is customary to express the fraction of density $\Omega_{i,0}$ multiplied by h^2 defined by

$$H_0 = 100 \times h \, \frac{\text{km}}{\text{sec Mpc}} . \tag{4.24}$$

Using $\Omega_{i,0}h^2$ instead of $\Omega_{i,0}$ one is immune to changes or errors in the measurement of H_0. In other words, measurements of the actual energy density $\rho_{i,0}$ can be converted into $\Omega_{i,0}h^2$ without assuming the value of any other cosmo parameter. This is a particularly important point since as of 2018, percent level measurement of H_0 present a 3-4 σ tension. In particular CMB/BAO measurements give $h = 67.6 \pm 0.6$ [1] while local measurements based on the distance ladder, which find $h = 73.24 \pm 1.74$ [41]. In this way the measurement of ρ_m is not contaminate by the error on H_0. Notice that $h^{-2} \simeq 2$.

Finally, using (P.5.1) age of the universe is then (see P.5.1)

$$t_{\text{age}} = \frac{1}{H_0} \int_0^1 \frac{da}{a} \left[\Omega_\Lambda + \Omega_m a^{-3} + \Omega_r a^{-4}\right]^{-1/2} . \tag{4.25}$$

Problems for lesson 4

P.4.1 Compute the particle horizon for a matter and radiation dominated universe of fixed age. Which one is larger? Interpret your answer.

P.4.2 (Mukhanov's book ex 1.9) By embedding a 3D sphere (pseudo-sphere) in a $(3+1)$D Euclidean (Lorentzian) space, verify that the metric of a 3D space of constant curvature can be written as

$$dl^2 = R^2 \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin\theta^2 d\phi^2\right)\right] , \tag{4.26}$$

where $R > 0$ and $k = 0, \pm 1$.

[28] Beware of different conventions. Sometimes $\Omega_a(t)$ is defined in terms of the time dependent critical energy density $3M_{\text{Pl}}^2 H^2$ and/or the time dependent density $\rho(a)$ and sometimes it is just its value today, which I indicate as $\Omega_{a,0}$ to avoid confusion.

Check for understanding of lesson 4

cfu.4.1 Discuss the difference between Doppler and cosmological redshift. You can follow the discussion in this note or read [23].

cfu.4.2 Does any of the distances introduced in this lesson (comoving, angular diameter and luminosity distances) correspond to geodesic distance? Devise a though experiment that would measure geodesic distance.

LESSON 5

Constituents of the universe

ref

We will now review the five main components of the universe: photons, baryons, neutrinos, dark matter and dark energy.

Only particles with a lifetime comparable with the age of the universe have a sizable density today. Within the standard model of particle physics, we have photons, protons and electrons. Free neutrons decay in about 15 minutes, but they can be stable when combined with protons to form the nuclei of atoms.

5.1 Photons

The density of *photons* can be derived straightforwardly from the temperature of the CMB, $T_{\rm CMB} = 2.72548 \pm 0.00057$ K [17, 18]. We know that the (dimensionless) chemical potential is small[29] (defined in Eq. (6.4)) $\mu < 6 \times 10^{-5}$, so we can use the exact Planck black body spectrum as in (6.13):

$$\rho_\gamma = 3p_\gamma = \frac{\pi^2}{15} T^4_{\rm CMB}\,, \tag{5.1}$$

where I used that a photon is a boson with two degrees of freedom $g_\gamma = 2$ (helicities ± 1). From the covariant conservation of energy we know that $\rho_\gamma \propto a^{-4}$ and therefore for photons $T\,a = {\rm const}$. Finally, one finds

$$\Omega_\gamma h^2 = 2.5 \times 10^{-5}\,, \quad \to \quad \Omega_\gamma \simeq 5 \times 10^{-5}\,. \tag{5.2}$$

5.2 Baryons

In particle physics, the word *baryons* indicates only protons and neutrons but in cosmological lingo it is customary to include electrons as well. The universe appears to be neutral as a whole, so we will assume as many electrons as protons[30]. With the prominent exception of neutrinos, all other hadrons and leptons are present in negligible amount because they decayed long ago. Big Bang Nucleosynthesis (BBN) makes predictions that are extremely well confirmed by observations: 75% Hydrogen (single proton with only traces of deuterium) and $24.5 \pm 0.004\%$ of Helium [4] (2 protons, 2 neutrons, with traces of ^{3}He). All other elements have negligible densities (subject of next lecture). There are three main ways to measure Ω_b:

- most baryons are in the *intergalactic medium*, rather than in stars, where $\Omega_b h^2 \simeq 0.02$.
- The light of distant *quasars* (quasi-stellar radio sources, very bright and very distant galaxies, $z \lesssim 7$; they cannot be resolved so "look like" stars) is absorbed during its propagation by the intervenient Hydrogen
- The oscillating patters in the *CMB power spectrum* is very sensitive to all cosmological parameters. $\Omega_b h^2$ changes the height of the acoustic peaks because it displaces the averages of the oscillations and changes the diffusion damping scale. Planck gives the imposingly tight constraint (see Table 3 of [1])

$$\Omega_b h^2 = 0.02225 \pm 0.00016 \to \Omega_b \simeq 0.05 \tag{5.3}$$

- The abundance of light elements produced during *Big Bang Nucleosysnthesis* (BBN) depends very sensitively the baryon density, leading the constraint $\Omega_b h^2 = 0.022 \pm 0.006$ [8]

As we will see, observations of the total matter density in the universe point to a much larger fraction than a few percent, so there must be some other type of non-baryonic matter[31].

[29] The bound come mostly from the instrument FIRAS on board of the COBE satellite [18], which reported in 1996, $\mu < 9 \times 10^{-5}$. Recently, the ground based experiment TRIS [20] has provided a mild improvement ($\mu < 6 \times 10^{-5}$) by decreasing the degeneracy with other parameters.

[30] cfu: *Is there a good explanation for this?*

[31] Remember that in the cosmology slang, "matter" means a component scaling approximately as $\rho_b \propto \rho_{DM} \propto a^{-3}$

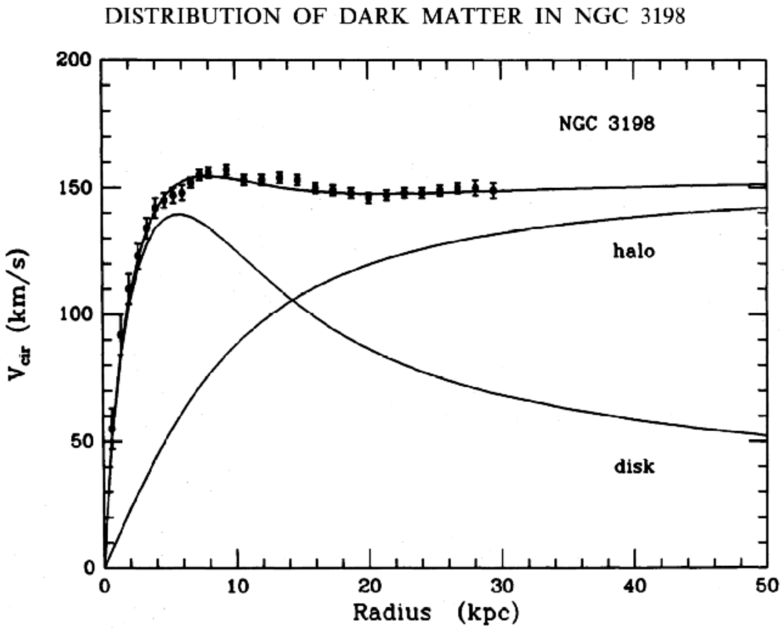


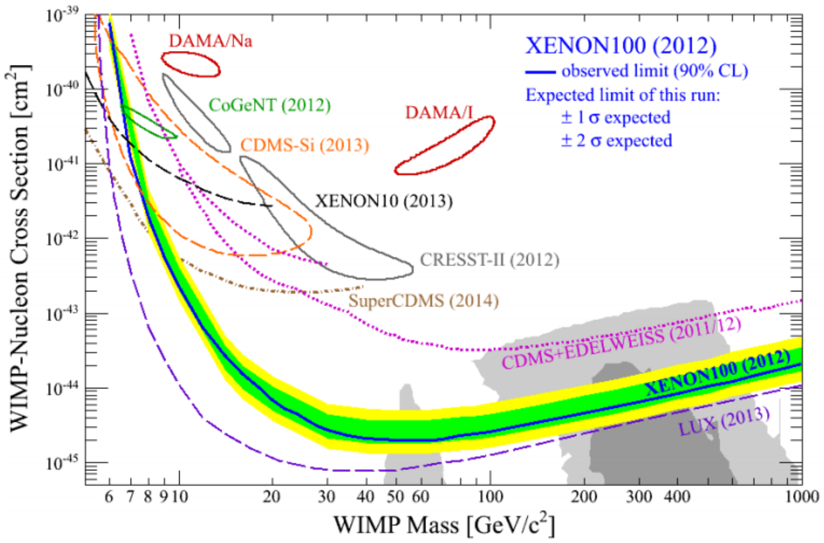
Figure 5: The plot shows the orbital velocity of start in the galaxy NGC 3198 as function of radius. The "disk" line shows what would be expected if all the matter in the galaxy were the baryons in the (flat) galactic disk. By adding a diffuse dark matter component, a.k.a. "halo", one can reconcile predictions with observations.

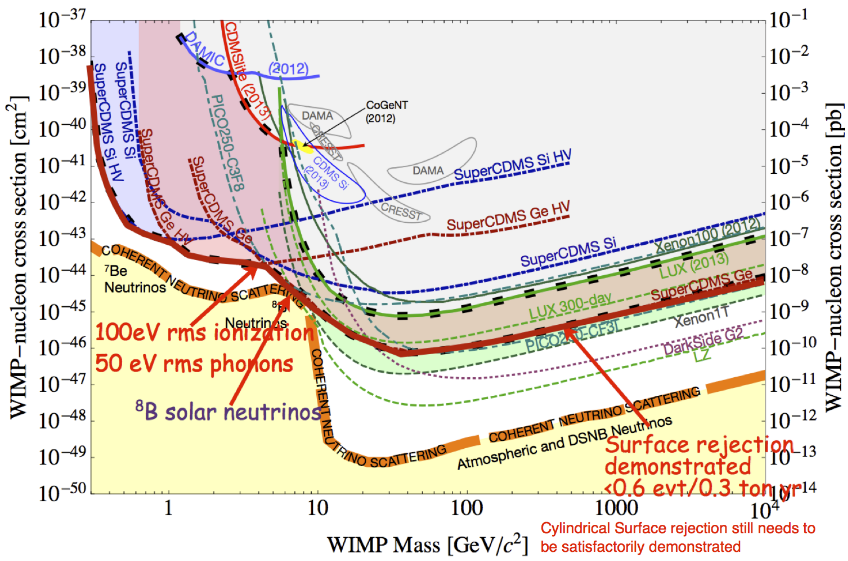
Dark Matter The evidence for a non-baryonic component or *Dark Matter* comes exclusively from gravitational physics, unlike that for baryons (which we observed using the light they emit). Evidence comes from all possible scales

- The Swiss astronomer Fritz Zwicky pointed this out already in the '30 that the orbits in the Coma *galaxy cluster* indicated the existence of "dunkle Materie". But much astrophysical uncertainty was present and few believed him at the time.
- As the American astronomer Vera Rubin pointed out in the '70, something similar can be seen for galaxies. As can be seen in Fig. 5, observations tell us that *galaxy rotation curves* flatten out rather than decaying to zero as it would be the case if the only matter present were stars and interstellar medium [homework P.5.2].
- Now we have probed all other bound objects, from groups of galaxies to superclusters. One can measure the *mass-to-light ratio* in many of these systems. This grow with scale and saturates at $\Omega_m \simeq 0.3 > 0.05$.
- The CMB comes from when baryons where ionized and therefore evolved very differently form DM. The CMB Temperature-Temperature power spectrum is sensitive to Ω_{DM}, e.g. through the relative height of odd and ever peaks and the size of the diffusion damping. The current bound from Planck is $\Omega_{DM} h^2 = 0.1198 \pm 0.0015$ leading to $\Omega_{DM} \simeq 0.0267$ [1].
- The matter power spectrum possesses small oscillations because the baryons were oscillating with the photons before decoupling, unlike DM. These *Baryon Acoustic Oscillations* pin down the ratio Ω_b/Ω_m giving a consistent measurement

5.3 Neutrinos

Neutrinos (see [14, 22, 26, 27] for a review) are the lightest fermions in the standard model and come in three families: ν_e, ν_μ and ν_τ. They carry no electric charge or "color" and interact weakly being part





of an SU(2) doublet together with each family of charged left-handed leptons, namely electron, muon or tau. These neutrino states have well-defined weak charge but they are not energy eigenstates. The linear relation between charge and energy eigenstates is

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \tag{5.4}$$

where $s_{ij} \equiv \sin\theta_{ij}$ and $c_{ij} \equiv \cos\theta_{ij}$. Hence, the free propagation of neutrinos is determined by three masses, three mixing angles θ_{ij} and one CP-violating phase δ. Neutrino oscillations imply that at least two neutrinos have non-zero mass (Nobel prize 2015) [31]:

$$\Delta m^2_{21} \;=\; (7.9^{+1.0}_{-0.8}) \times 10^{-5}\ \text{eV}^2 \qquad |\Delta m^2_{31}| = (2.2^{+1.1}_{-0.8}) \times 10^{-3}\ \text{eV}^2 \tag{5.5}$$

What these measurements cannot determine is the overall scale of neutrino masses as the sign of Δm^2_{31}. The latter uncertainty implies that there are two possible mass ordering for the three eigenstates, as shown in the left panel of figure6. At present, the tightest bounds on the *sum of neutrino masses* come from cosmology. Combining CMB anisotropies with Baryon Acoustic Oscillations (BAO) gives (see 6.4 of [1])

$$\sum_{i=1}^{3} m_i < 0.17\,\text{eV}\,. \tag{5.6}$$

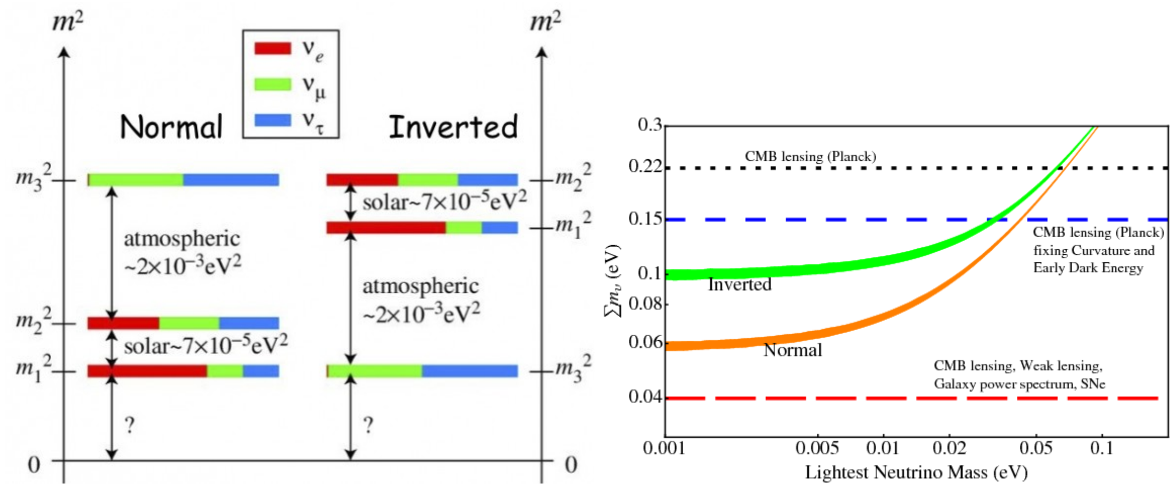


Figure 6: Left: the normal and inverted scenarios for the neutrino hierarchy. Right: the total neutrino mass as function of the yet unknown mass of the lightest neutrino. The current bound are shown in the black dotted and blue dashed lines, while the red long-dashed line represent the expected future sensitivity. This in particular shows that the sum of neutrino masses will be detected in the near future.

There is no explanation for the neutrino mass in the standard model and various models have been proposed. It is also still not known whether neutrinos are their own antiparticle (namely they are Majorana fermions) or not (Dirac fermions like the electron-positron). With the large improvement in the precision of cosmological observations, we have now many different probes that will be able to detect neutrino masses and determine (see righthand panel of Fig. 6) the correct hierarchy in the next 5 to 10 years!

Unlike their mass, the abundance of cosmological neutrino (sometimes called CNB or CνB for Cosmological Neutrino Background) has been observed via CMB anisotropies. The actual constraint is often quoted in terms of the effective number of neutrinos N_{eff}. Standard model predicts $N_{eff} = 3.04$ [32], which is fully compatible with the current CMB constraints $N_{eff} = 3.04 \pm 0.18$ (see 6.4 of [1]). Let us see what this parameter means. Three periods characterize the evolution of cosmological neutrinos:

- Thermal equilibrium with SM particles at energies around a few MeV[32]. Neutrinos are very relativistic (MeV$\gg$ 0.17eV) and obey a FD distribution with $\mu = 0$ and massless dispersion relation
- Neutrino decoupling before electron-positron annihilation. As long as neutrinos are relativistic ($z \gg 500$), the neutrino temperature is

$$T(a) = T_{dec}\frac{a_{dec}}{a} \tag{5.7}$$

- Neutrinos became non-relativistic at late times ($z < 500$) and start clustering

We compute the temperature of neutrinos by relating it to that of photons, namely T_{CMB}. An order one effect is the extra energy that photons receive after electron-positron annihilation ($e^+ + e^- \to \gamma + \gamma$), which the neutrinos do not receive because they are already decoupled. Covariant conservation of entropy[33] in an FLRW universe implies $\partial_t(a^3 s) = 0$. Before e+e- annihilation and neutrino decoupling, the total entropy is dominated by relativistic species (see Section P.9.4), and was calculate in (6.15). For us

$$\begin{aligned} s_1 &= \frac{2\pi^2}{45}T_1^3\left[g_{boson} + \frac{7}{8}g_{fermions}\right] &(5.8)\\ &= \frac{2}{3}\frac{\pi^2}{15}T_1^3\left[2 + \frac{7}{8}2\times(1+1+3)\right], &(5.9)\end{aligned}$$

[32] Different species decouple at slightly different times. Neglecting mass oscillations, one finds $T(\nu_e) \simeq 2.4$MeV and $T(\nu_{\mu\tau}) \simeq 3.7$MeV [22]

[33] Electron position annihilation proceeds in states of equilibrium, since it could be reverse by re-contracting and heating up the universe around the transition temperature. Therefore the total entropy is conserved

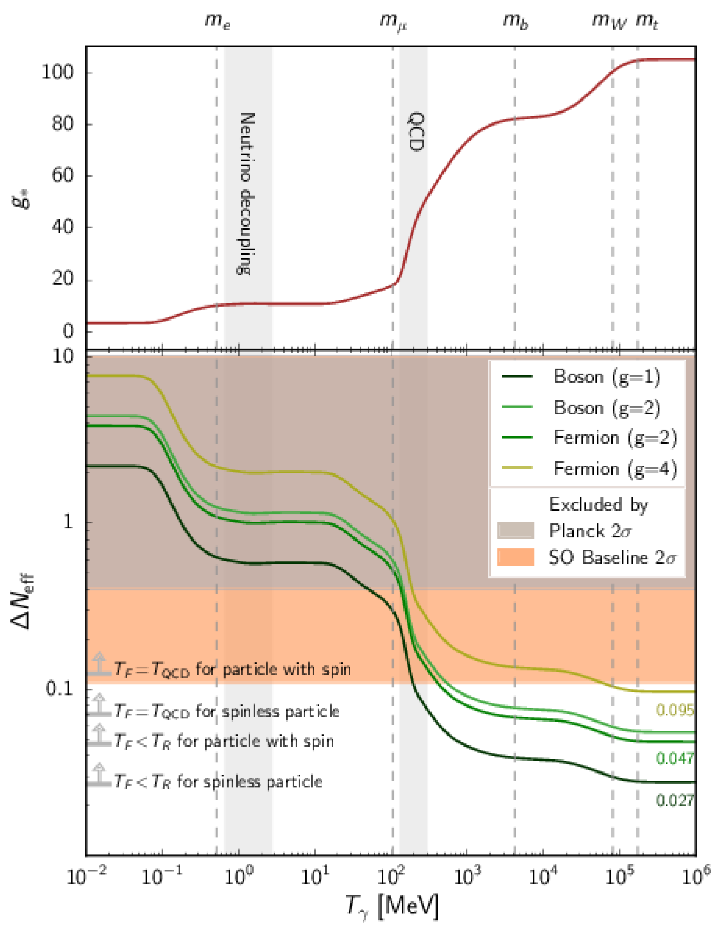


Figure 7: Taken from [2]. The plot shows $\Delta N_{\rm eff} \equiv N_{\rm eff} - 3.04$ and the number of degrees of freedom g_* as function of the decoupling temperature T_γ for various types of particles.

where the bosons are just the two polarizations of the photon, and the fermions are the two helicities of e^-, e^+ and the three neutrinos[34]. Then neutrinos decouple and their temperature redshifts such that Ta is constant, so they maintain the same temperature as photons until e+ e- annihilation at around 0.5 MeV. After the annihilation, the total entropy is given by[35]

$$s_2 = \frac{2}{3}\frac{\pi^2}{15}\left[2T_\gamma^3 + \frac{7}{8}2 \times 3T_\nu^3\right] , \tag{5.10}$$

where now we accounted for the fact that the neutrinos are not in equilibrium with the photons and so could and indeed have a different temperature T_ν. We can use the conservation of entropy $a_1^3 s_1 = a_2^3 s_2$ and $a_1 T_1 = a_2 T_\nu$ to find

$$\boxed{T_\nu = T_\gamma \left(\frac{4}{11}\right)^{1/3}} \qquad \Rightarrow \qquad T_\nu(a_0) \simeq 1.96\,\mathrm{K} \simeq 1.7 \times 10^{-4}\,\mathrm{eV} . \tag{5.11}$$

So neutrinos are a bit colder than photons at any time after e+ e- annihilation. Notice that this does not depend on whether neutrinos are Dirac or Majorana.

To compute the neutrino energy fraction today Ω_ν, we have to account for their mass. The precise calculation can only be done numerically, but there are two interesting analytical limits. First let us

[34] Notice that protons and neutrons are non-relativistic (GeV≫MeV). The baryon to photon number ratio is $n_b/n_\gamma \sim 10^{-9}$ and so baryons lead to a tiny contribution to the total entropy density. Electron and positron instead are quasi relativistic

[35] The number density of surviving electrons is about $n_e \sim 10^{-9} n_\gamma$ (same as for baryons), so they can be neglected in the entropy.

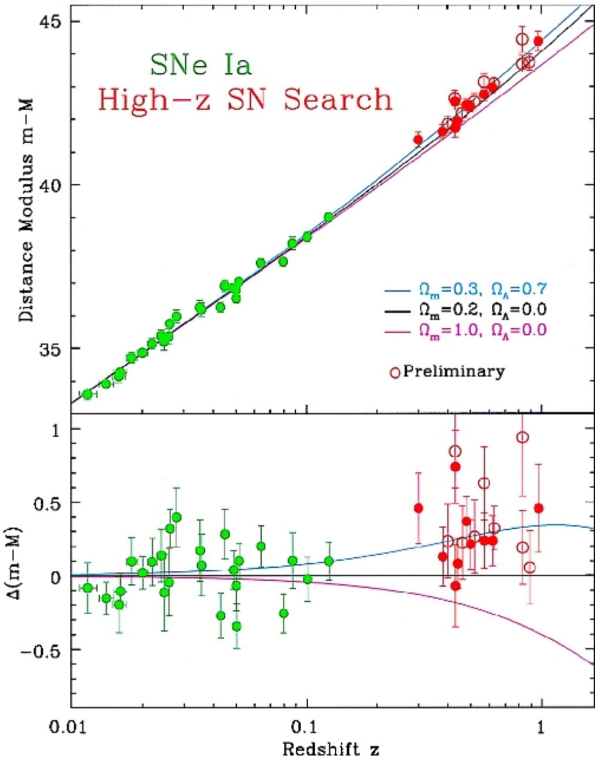


Figure 8: Left: Dodelson's fig 1.7. Right: the same plot as required in exercise 16 chapter 2 with two flat universes, one with $\Omega_\Lambda = 0.7$ (continuous blue line) and another with $\Omega_\Lambda = 0$ (dashed orange line).

assume the neutrinos are massless. The integral of the FD distributions is smaller than that over the BE distribution by a factor of 7/8 so we find

$$\rho_\nu = \rho_\gamma \, 3 \, \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} , \quad h^2 \Omega_\nu = 1.7 \times 10^{-5} \quad (m_\nu = 0) \, . \tag{5.12}$$

At early times, when neutrinos are still relativistic ($z \gg 500$), the total radiation energy density ρ_r is given by

$$\rho_r = \rho_\gamma \left[1 + N_{\rm eff} \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} \right] , \tag{5.13}$$

where $N_{\rm eff}$ quantifies the number of relativistic species in the universe besides the photons. In the standard model, $N_{\rm eff} = 3.04$ for the three neutrino species. The slight deviation from 3 comes from the fact that neutrinos still have some small interaction with the SM at $e^+ - e^-$ annihilation and so receive tiny bit of heating as well. In analyzing the data, one can treat $N_{\rm eff}$ as a free parameter to test for deviations from the standard model. Currently CMB data gives the constraint $N_{\rm eff} = 3.04 \pm 0.18$ [1], implying a detection of a Cosmic neutrino Background CνB. Sensitivity to $N_{\rm eff}$ is expected to improve by a factor of three in the next ten years with the Simons Observatory (SO) even further with CMB Stage 4 (S4) [2]. This could detect or exclude any particle that has *ever* been in thermal equilibrium with the Standard Model (see Fig. 7).

If neutrinos are massive, when they are fully non-relativistic their energy density is simply given by

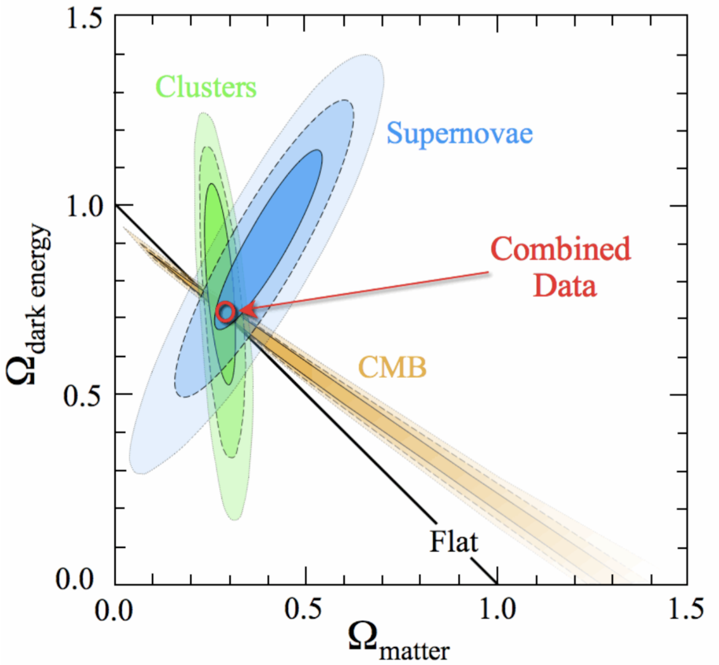


Figure 9: The concordance or standard model of cosmology. Cluster counts, supernovae and the CMB agree on a flat ($K = 0$ accelerated universe, dominated by Dark Energy)

$\rho_\nu = m_\nu n_\nu$, with n_ν their conserved number density defined by Eq. (**??**). One then finds (see P.5.4)

$$n_\nu = \frac{3}{11} n_\gamma = \frac{6\zeta(3)}{11\pi^2} T_\gamma^3 \simeq 113\,\mathrm{cm}^{-3}\,, \tag{5.14}$$

$$\Omega_\nu = \frac{\rho_\nu}{\rho_{cr}} = \frac{\sum_{i=1}^{3} m_i}{94\,h^2\,\mathrm{eV}} \quad (m_\nu \neq 0)\,. \tag{5.15}$$

Neutrinos were originally proposed to explain the entirety of dark matter but they are too light and one finds $\Omega_\nu \leq 0.4\%$. Nevertheless, neutrinos do cluster[36] and they do produce small effects on structure formation. A large number of experiments aims at detecting these effects in the next decade.

5.4 Dark Energy

In the late 90's evidence began to accumulate that $\ddot{a}(t_0) > 0$, i.e. the current expansion of the universe is accelerating. The discovery was announced by two groups: High-Z Supernova Search Team [40] and the Supernova Cosmology Project [37], both of which got the Nobel prize in 2011. Supernovae of Type 1a are exploding stars whose progenitor is a small and compact start called a white dwarf in a binary system (i.e. orbiting another, usually larger star). SN1a are standard candles so their intrinsic luminosity should be approximately the same (corrected for some dust absorption and some "unknown" environmental dependence). We can calibrate nearby SN1A and hence know the intrinsic luminosity L. So, if we measure a SN1A, we can deduce its luminosity distance d_L, since we know L and measure the flux l in (4.16). In addition, the redshift of each supernova can be measured from emission and absorption lines in its spectrum. The resulting luminosity distance $d_L(z) = \chi(z)(1+z)$ or apparent magnitude $m - M$ as function of redshift are in Fig. 8. This is somewhat analogous to the classic Hubble diagram in Fig. 1, with the remarkable difference that is extends to much further objects ($z \sim 1$ so a few Gpc as opposed to the few Mpc in Hubble's diagram). One sees that SN appear fainter in our universe than they should appear in a matter dominated universe. Introducing a cosmological constant, the measurements agree with predictions. Also, this estimate of the accelerated expansion agrees beautifully with CMB and cluster counts. This is known as the concordance model or LCDM for Λ Cold Dark Matter (see Fig. 9).

[36]Clustering means to get denser or sparser around over or underdensities. Very relativistic particles, such as for example photons, do not cluster because they cannot be captured by the gravitational potential of even the largest clusters of galaxies.

From the acceleration equation (3.52) we know that $\ddot{a} > 0$ implies $\rho + 3P < 0$ or equivalently $w < -1/3$. Neither matter nor radiation can produce this effect since they both obey the Strong Energy Condition, i.e. for ever future pointing time-like vector X^μ

$$\left(T_{\mu\nu} - \frac{1}{2}T^\lambda{}_\lambda g_{\mu\nu}\right) X^\mu X^\nu \geq 0 \quad \Rightarrow \quad \rho + 3p \geq 0 \quad \text{(SEC)}\,. \tag{5.16}$$

We are then forced to either change the framework within which we interpret the data (e.g. change the laws of gravity of the FLRW metric) or introduce a new constituent of the universe: *Dark Energy*. A detailed study of the data shows that for Dark Energy to produce the accelerated expansion of the universe $\ddot{a} > 0$ we need

$$p_{DE} = -\rho_{DE}\,(1 \pm 0.05) \quad \text{and} \quad \Omega_{DE,0} \simeq 0.7 \tag{5.17}$$

Dark Energy is the politically correct and over-encompassing name for all the proposed theories of late cosmic acceleration. The *cosmological constant*, *quintessence* and *modified gravity* are among the most investigated scenarios for Dark Energy.

Let us briefly discuss the naively most conservative solution to late cosmic acceleration: a cosmological constant. Diffeomorphism invariance of the EE allows for an additional constant term

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = M_{\rm Pl}^{-2}\left(T_{\mu\nu} - \Lambda_{cc}g_{\mu\nu}\right)\,, \tag{5.18}$$

which had originally been introduced by Einstein to find a static universe (see P.5.5). Interpreting $-\Lambda_{cc}g_{\mu\nu}$ as the energy-momentum tensor of the cosmological constant, we deduce $p_{cc} = -\rho_{cc}$ and therefore[37] $w_{cc} = -1$. Notice that as the universe expands or contract the energy density remains constant. Because of this the cosmological constant is also called *vacuum energy* meaning that $\rho_{cc} = \Lambda$ is an energy density associated with "empty" spacetime itself. Equivalently, the most general action compatible with the symmetries is

$$S = \int d^4x\sqrt{-g}\left[\Lambda_{cc} + \frac{M_{\rm Pl}^2}{2}R + \mathcal{O}(R^2)\right]\,, \tag{5.19}$$

where additional terms such as R^2 or R^3 have more spacetime derivatives. In my conventions the spacetime constant Λ_{cc} has dimension of an energy density $[\Lambda_{cc}] = E^4$. What can we say about the value of Λ_{cc}?

Consider General Relativity (GR) as and Effective Field Theory[38] (EFT). Since $[R] = E^2$, the theory is not renormalisable in the traditional sense, i.e. at every order in perturbation theory new operators need to be introduced with increasing number of fields and derivatives (i.e. higher and higher dimesion) to cancel new UV divergences. Yet, like every EFT, at energies E well below the naive cutoff $\Lambda_{\rm cutoff} \sim M_{\rm Pl}$, for experiment with some finite precision ϵ, there is just a finite number of counterterms needed to compute finite, renormalized observables. Naively, for predictions at some scale $E \ll \Lambda_{\rm cutoff}$ we need only operators of dimension l where $(E/\Lambda_{\rm cutoff})^l \geq \epsilon$. So the theory is predictive as long as we can make independent measurement to impose all renoramalization conditions on all operators of dimension l or less. We can then safely quantise gravity perturbatively, around some fixed (classical) background such as FLRW. When we couple GR to a given model of particle physics, the additional dynamics might introduce strong coupling at lower energies than $M_{\rm Pl}$. Since we have successfully tested the standard model of particle physics at accelerators, we conservatively assume TeV$< \Lambda_{\rm cutoff} < M_{\rm Pl}$. Here comes the key point. In a *natural* EFT's, every coupling constant is expected to be given by appropriate powers of the cutoff of the theory. For exampleΛ is expected to be the cutoff of the theory to the fourth power, and so

$$\Lambda_{cc} \simeq \Lambda_{\rm cutoff}^4 > \text{TeV}^4 \quad \text{(natural EFT expectation!?)}\,, \tag{5.20}$$

But the late time cosmic acceleration is an Infra-Red (IR) effect as compared with typical particle physics scales. Late acceleration is associated with an energy density in the universe of order

$$3H_0^2 M_{\rm Pl}^2 \sim (10^{-3} \times \text{eV})^4 \ll \text{TeV}^4 < \Lambda_{\rm cutoff}^4\,. \tag{5.21}$$

[37]cfu: *What is the critical w for which change from accelerated to decelerated expansion? Look back at the acceleration equation Eq. (3.52)*

[38]For an introductory discussion of EFT's see e.g. [39]

So the expectation based on naturalness are at least wrong by a factor of $(10^{15})^4$. The above considerations about the cosmological constant are usually summarised in terms of two conceptually distinct problems:

- The cosmological constant naturalness problems or why don't we observe a large contribution to the universe energy budget of order $\Lambda^4_{\rm cutoff} > \mathrm{TeV}^4$?
- The cosmological constant fine tuning problem or how does the tiny dimensionless number $\Lambda^4_{\rm cutoff}/M^2_{\rm Pl}H^2_0 > 10^{60}$ emerge from the laws of nature?

Many models have been constructed to address these issues over the years, but there is no clear favourite so far. An ambitious observational program is underway to test many of these theories. For more details see e.g. [9, 43].

Problems for lesson 5

P.5.1 Compute the age of the universe using for the numerical value of the cosmological parameter, the latest result from the Planck satellite [arXiv:1502.01589]. Compare with the quoted result in the same paper. [Hint: You will have to compute one integral numerically]

P.5.2 Compute the galaxy rotation curve, namely the velocity v as function of radius R, assuming that there is only baryonic matter (stars and interstellar gas, but no dark matter). You can use the Newtonian approximation and assume a Gaussian baryonic distribution $\rho(R) = \rho_0 e^{-(R/R_s)^2}$ where R_s is typically of order a few kpc, e.g. $R_s \sim 4\,\mathrm{kpc}$ for our Milky way. Notice that the distribution of luminous matter can be deduced from the luminosity of the galaxy as function of radius. The Gaussian profile above reproduces only qualitative the exponential decay at large radius. Qualitatively compare your result with some actual data (e.g. google image "galaxy rotation curves").

P.5.3 (Dodelson's Exercises 17 ch.2) Express the entropy density s as function of temperature T for massless bosons and fermions, assuming equilibrium and zero chemical potential. Neglecting chemical potential, show that a particle of mass m in equilibrium at $T \ll m$ gives an exponentially small contribution to the entropy $s \propto e^{-m/T}$.

P.5.4 (Dodelson's Exercises 18 ch.2) Show that the number density of one generation of neutrinos and anti-neutrinos in the universe today is approximately

$$n_\nu = \frac{3}{11} n_\gamma \sim 100\,\mathrm{cm}^{-3}\,. \tag{5.22}$$

P.5.5 Einstein originally introduced a cosmological constant in order to maintain a static universe. Find out how he was proposing to realize this. Consider a universe with matter, radiation, a cosmological constant defined by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \frac{1}{M^2_{\rm Pl}} T_{\mu\nu}\,, \tag{5.23}$$

and curvature K, so that

$$H^2 + \frac{K}{a^2} = \frac{1}{3M^2_{\rm Pl}}\left(\rho_M + \rho_R + \rho_\Lambda\right)\,, \tag{5.24}$$

with $\rho_\Lambda \equiv M^2_{\rm Pl}\Lambda$. Find the special value $\bar{a}$ of a and of Λ such that the universe is static. Is this static solution stable under perturbations away from $\bar{a}$?

P.5.6 Consider two supernovae, one with apparent magnitude $m = 24.3$ at $z = 0.83$ and one with $m = 16.08$ at $z = 0.026$. Neglecting error bars and assuming a flat universe with just matter and a cosmological constant, determine the preferred value of Ω_Λ.

Check for understanding of lesson 5

cfu.5.1 Imagine the neutrinos interacted much more strongly with electrons and protons, say an interaction rate $\Gamma = 10^3 \times \Gamma_{SM}$ where Γ_{SM} is the standard model value. In this hypothetical situation, would the neutrinos be hotter or colder today as compared with the real universe?

cfu.5.2 How do Dark Matter and Dark Energy need to interact or not interact to fit observations? Weak, strong, elctromagnetic or gravitational interactions?

cfu.5.3 Can cosmic neutrinos be the whole of Dark Matter or Dark Energy? Why?

LESSON 6

Thermal history

ref

So far we have learned that our universe was radiation dominated up to $z \simeq 3300$, then matter dominated until $z \simeq 0.4$ and Dark Energy dominated since. In this lesson, we discuss the thermal history of the homogeneous universe, which relies on equilibrium thermodynamics. Then we develop the formalism to statistically describe out of equilibrium processes using the Boltzmann equations. In the next lecture we will use this to study big bang nucleosynthesis (BBN), Dark Matter decoupling and recombination.

6.1 Relativistic kinetic theory

In a many particle system, one can also derive the energy-momentum tensor from *relativistic kinetic theory*. Consider the phase space density $f(x,p,t)$ for particles of mass m, defined as the infinitesimal probability $d\text{Prob}$ for finding a particle at position $\mathbf{x}$ with momentum $\mathbf{P}$ at time t by

$$d\text{Prob} = f(\mathbf{x},\mathbf{P},t)\prod_{i,j} d^3x^i d^3P_j\,. \tag{6.1}$$

The energy-momentum tensor and number density for a species a of particles are then

$$T_a^{\mu\nu}(\mathbf{x},t) = \frac{g_a}{\sqrt{-\det g}}\int \frac{\prod_k^3 dP_k}{(2\pi)^3 P^0} P^\mu P^\nu f_a(\mathbf{x},\mathbf{P},t)\,, \tag{6.2}$$

$$n_a(\mathbf{x},t) = \frac{g_a}{\sqrt{-\det g}}\int \frac{\prod_k^3 dP_k}{(2\pi)^3} f_a(\mathbf{x},\mathbf{P},t)\,, \tag{6.3}$$

where g_i is the degeneracy of the one-particle state, equivalently the number of propagating degrees of freedom. For example, $g=2$ for massless vectors such as the photon or massless tensors such as the graviton (only helicities ± 1 and ± 2 respectively); $g=2$ for a Dirac fermion such as the electron e^-, the positron e^+ or the proton p^+ (helicities $\pm 1/2$); $g=1$ for a Weyl or Majorana fermion such as the neutrino and its antiparticle[39].

It is convenient to adapt the very general integrals in (6.2) to the case of most relevance in cosmology. First, let us then consider particles that are in equilibrium, and therefore obey Bose-Einstein or Fermi-Dirac statistics[40]

$$f_a(\mathbf{x},\mathbf{P},t) = f_{BE,FD}(\mathbf{x},|\mathbf{P}|,t) = \left[e^{(P^0-\mu)/T} \mp 1\right]^{-1}\,, \tag{6.4}$$

where $P^0 = \sqrt{m^2 + P^i g_{ij} P^j}$ and the spacetime dependence appears in the *chemical potential* $\mu = \mu(\mathbf{x},t)$ (defined to be dimensionless) and the temperature $T = T(\mathbf{x},t)$. Second, let us restrict ourselves to a flat FLRW universe, (3.37), for which $\sqrt{-g} = a^3$. Third, by homogeneity and isotropy, the only non-vanishing components of the energy momentum tensor are $T^0{}_0 = \rho$ and $T^i{}_i = 3p$. Changing integration variable from the comoving momentum P^i to the physical three-momentum

$$\mathbf{q} \equiv \sqrt{a^2 P^j P^i \delta_{ij}} \quad \Rightarrow \quad dP_k = a d\mathbf{q}\,, \tag{6.5}$$

all factors of a cancel out. Then, using spherical coordinates, the angular integrations in (6.2) simply give a factor of 4π. Finally, we find

$$\rho_a(\mathbf{x},t) = \frac{g_a}{2\pi^2}\int dq\, q^2 \frac{E}{e^{(E-\mu)/T} \mp 1}\,, \tag{6.6}$$

$$p_a(\mathbf{x},t) = \frac{g_a}{2\pi^2}\int dq\, q^2 \frac{q^2}{3E}\frac{1}{e^{(E-\mu)/T} \mp 1}\,, \tag{6.7}$$

$$n_a(\mathbf{x},t) = \frac{g_a}{2\pi^2}\int dq\, q^2 \frac{1}{e^{E(q)/T} \mp 1}\,, \tag{6.8}$$

for bosons and fermions respectively, with $E(q) = \sqrt{m^2+q^2}$.

[39] It is not known whether the neutrino is a Weyl or a Majorana particle. Either way the final counting is the same: the 4 real components for a Majorana spinor can be written in terms of the 2 complex components for a Weyl spinor. A Weyl spinor is a chiral particle (e.g. left-handed for neutrinos), with an antiparticle of opposite chirality (the right-handed anti-neutrino). A Majorana particle instead has both chiralities and is its own anti-particle.

[40] cfu: *How do you get the right sign in the denominator of $f_{BE,FD}$? Remember the exclusion principle for Fermions which implies $f_{FD} < 1$.*

Relativistic particles For $T \gg m$, these integrals are mostly supported around $q \simeq 3T$ and so at high temperature we can approximate $E(q) = \sqrt{m^2 + q^2} \simeq q$ up to corrections of order (m/T). At this order $p^2/3E = E/3$ and so $\rho = 3p$. Performing the integrals [41] above one finds

$$\rho_a = 3p_a = g_a \frac{3}{\pi^2} T^4 \left[\pm \mathrm{Li}_4(\pm e^{\mu/T}) \right] , \tag{6.11}$$

$$n_a = g_a \frac{1}{\pi^2} T^3 \left[\pm \mathrm{Li}_3(\pm e^{\mu/T}) \right] , \tag{6.12}$$

for bosons and fermions respectively, where $\mathrm{Li}_n(z)$ is the polylogarithm of z at order n. The chemical potentials are small for all known particles and almost all times, so we can simplify these expressions in the limit $\mu \ll T$. The polylogarithms can be evaluated analytically and one finds

$$\boxed{\rho_a = 3p_a = g_a \frac{\pi^2}{30} T^4 \left\{ \begin{array}{ll} 1 & \text{(relativistic bosons)} \\ \frac{7}{8} & \text{(relativistic fermions)} \end{array} \right.} , \tag{6.13}$$

as well as

$$n_a \quad = \quad g_a \frac{\zeta(3)}{\pi^2} T^3 \left\{ \begin{array}{ll} 1 & \text{(relativistic bosons)} \\ \frac{3}{4} & \text{(relativistic fermions)} \end{array} \right. , \tag{6.14}$$

where the Riemann zeta-function is approximately $\zeta(3) \simeq 1.2$. We can also compute the entropy density (always neglecting μ)

$$s = \frac{\rho + p}{T} = g_a \frac{2\pi^2}{45} T^3 \left\{ \begin{array}{ll} 1 & \text{(relativistic bosons)} \\ \frac{7}{8} & \text{(relativistic fermions)} \end{array} \right. . \tag{6.15}$$

Non-relativistic particles In the opposite limit, at low temperatures $m - \mu \gg T$ both quantum statistics reduce to a Boltzmann distribution since

$$e^{(\sqrt{m^2+q^2}-\mu)/T} > e^{(m-\mu)/T} \gg 1 \, . \tag{6.16}$$

Now the integral is mostly supported around $q \simeq \sqrt{Tm} \ll m$. If we also assume $m \gg T$, we can approximate $\sqrt{m^2 + q^2} \simeq m + q^2/(2m)$ everywhere, up to correction of order $T/m \ll 1$. Then the integrals can be done analytically and the result is

$$n_a \quad = \quad g_a \left(\frac{mT}{2\pi} \right)^{3/2} e^{(\mu-m)/T} \, , \tag{6.17}$$

$$\rho_a \quad = \quad g_a \left(\frac{mT}{2\pi} \right)^{3/2} e^{(\mu-m)/T} \left(m + \frac{3}{2} T \right) = n_a \left(m + \frac{3}{2} T \right) , \tag{6.18}$$

$$p_a \quad = \quad g_a \left(\frac{mT}{2\pi} \right)^{3/2} e^{(\mu-m)/T} T = n_a T \tag{6.19}$$

$$s_a \quad = \quad \frac{\rho_a + p_a}{T} = \, . \tag{6.20}$$

Notice that, if relativistic and non-relativistic particles are in thermal equilibrium, i.e. at the same temperature, and $\mu \ll T$, then

$$\frac{\rho_{\text{non-rel}}}{\rho_{\text{rel}}} \propto e^{-m/T} \left(\frac{T}{m} \right)^{5/2} \ll 1 \, . \tag{6.21}$$

We conclude that in thermodynamical equilibrium, particles become irrelevant for the total energy, pressure and entropy budge as soon as they become non-relativistic. It convenient to have a simple

[41] Here we have used the master integral

$$\int_0^\infty dy \, \frac{y^n}{e^{y-z} - \eta} = \frac{1}{\eta} \Gamma(n+1) \mathrm{Li}_{n+1}(e^z \eta) \tag{6.9}$$

where the polylogarithm is the generalisation of the logarithm in the sense that $\mathrm{Li}_1(z) = -\log(1-z)$ and

$$\mathrm{Li}_{n+1}(z) = \int_0^z \frac{\mathrm{Li}_n(z')}{z'} dz' \, . \tag{6.10}$$

expression for the radiation energy density. Let us introduce the effective number of bosonic degrees of freedom g_* defined as

$$\boxed{\rho = g_* \frac{\pi^2}{30} T^4 \,, \quad \text{with} \quad g_* \equiv g_{\text{bosons}} + \frac{7}{8} g_{\text{fermions}}} \,. \tag{6.22}$$

Box 6.1 Time-temperature relation Various quantities can be used to parameterize time: time (proper t or comoving τ), redshift z, the Hubble scale $H \equiv \dot{a}/a$, the particle horizon, the scale factor a, temperature, etc. A summary of the conversion is provided in table Tab. 1. A useful relation is that between cosmic time t and temperature T (see P.6.1). Let us recall the continuity equation for radiation, $w = 1/3$ and use $\rho \propto T^4$ from (6.13)

$$0 = \dot{\rho} + 4H\rho \propto \left(\dot{T} + HT \right) . \tag{6.23}$$

For a number g_* of relativistic species in thermodynamic equilibrium at temperature T the Friedmann equation gives

$$H = \sqrt{\frac{\rho}{3M_{\text{Pl}}}} = \sqrt{g_* \frac{\pi^2}{90}} \frac{T^2}{M_{\text{Pl}}} . \tag{6.24}$$

We can then solve the o.d.e.

$$\dot{T} = -T^3 \frac{\pi}{M_{\text{Pl}}} \sqrt{\frac{g_*}{90}} \tag{6.25}$$

and find $T(t)$ and its inverse

$$T(t) = \left(\frac{5}{2g_*} \right)^{1/4} \sqrt{\frac{3M_{\text{Pl}}}{\pi t}} \quad \Rightarrow \quad t = \sqrt{\frac{5}{2g_*}} \frac{3M_{\text{Pl}}}{\pi T^2} \quad \text{(radiation domination)} . \tag{6.26}$$

6.2 Thermal history

Let us now review the most important events and scales in chronological order (see Fig. 10)

- $T \sim 10^{18}$ GeV, approximately 10^{-43} sec: the perturbative quantum description of GR breaks down and the theory needs a Ultra-Violet (UV) completion. For example, new, unknown degrees of freedom could appear at or before this scale. This happens e.g. in *String Theory*, where higher spin particles become dynamical at the string scale[42] $M_s \lesssim M_{\text{Pl}}$. Alternatively the theory could become strongly coupled and we don't know what happens. It has been conjectured that GR might possess a UV-fixed point, where all coupling constants of the theory (including all higher dimension operators) have vanishing beta functions. This line of investigation goes under the name of *Asymptotic safely*. Many other approach to tackle non-perturbative quantum gravity have been proposed.

- $H \sim 10^3$ GeV - 10^{13} GeV, a conjectured phase of accelerated expansion called cosmological *inflation* seeds the primordial perturbation that later will give rise to the structure in the universe and eventually to us. The energy scale of this process is one of the most uncertain scales in physics. During inflation, the universe is cold and empty[43], the abundance of any standard model species is exponentially suppressed in time by the fast expansion $a \sim e^{Ht}$, with H approximately constant. The universe expands by at least a factor of approximately[44] $a_f/a_i \sim e^{60} \sim 10^{26}$. Inflation ends as the degree of freedom driving, some form or scalar condensate known as the *inflaton*, breaks up into particles, which in turns decay into standard model fields in a process called *reheating*. In the simplest and most standard paradigm, this final states consists of a hot ($T >$TeV) thermalized soup of SM particles. The *hot big bang* starts here.

- $T >$100 GeV, an asymmetry in baryon number is created by some, yet unknown, non-equilibrium (P and CP violating []) process called *baryogenesis*. As all quarks annihilate with anti-quarks, and

[42] The value of the string scale is of course unknown and depends on the details of the compactification from 10 (or 11) down to 4 dimensions.
[43] Not unlike some places in Canada.
[44] The exact number of factors of e, namely $N \equiv \ln(a_f/a_i)$, aka *efoldings*, is not known. Many inflationary models have $40 < N < 60$, while data constraints $N > 20$ [].

only a part in a million of the baryonic matter in the universe survives. This will eventually form all atoms in the universe.

- $T \sim 100$ GeV - 10^3 GeV, the *electroweak symmetry* of the standard model $SU(2) \times U_Y(1)$ is broken via the Brout-Englert-Higgs mechanism down to the abelian $U(1)$ gauge symmetry that we call electromagnetism. The details of this phase transition depend crucially on the properties of the Higgs particle and of the spectrum of the standard model, which are being currently probed at particle accelerators such as the Large Hadron Collider at CERN. All SM fermions (quarks and leptons) as well as the $W^{\pm}$ and Z^0 vector bosons acquire a mass proportional to the vacuum expectation value (vev) of the Higgs field.

- $T \sim 200$ MeV the free quarks and gluons become confined as the coupling of the strong interactions becomes of order one. Because of its non-perturbative nature, the details of this *QCD phase transition* leading to confinement are still not fully understood. As the temperature decreases below the mass of the lightest mesons (the pions, $\pi^{\pm,0}$ whose mass is protected by the approximate global isospin symmetry), all quarks and gluons in the universe become confined inside protons and neutrons, which obey an thermal distribution.

- $T \sim 1 - 3$ MeV, 0.2 sec: neutrinos fall out of equilibrium as their weak interaction rate becomes smaller than the expansion of the universe (different neutrino flavor, $\nu_{e,\mu,\tau}$ decouple at slightly different energies). From this moment onward, neutrinos couple only gravitationally and mostly free stream across the universe.

- $T \sim 1$ MeV: the neutrons fall out of thermal equilibrium and their abundance freezes out (up to some decaying rate which, on the time scale of the problem, produces only an order 10% effect). The ratio of protons to neutrons in the universe is approximately fixed by this process.

- $T \sim 0.5$ MeV, 5 sec: *electron-positrons annihilation.* As the temperature drops below the electron mass 0.5 MeV, the process of electron-positron production becomes very rare and all positrons annihilate with electrons. As we observe an electrically neutral universe, a number of electron survive equal to the number of protons. As discussed around Eq. (5.11), this process releases energy into the photons, which therefore become hotter than the neutrinos (which had decoupled early).

- $T \sim 0.07$ MeV, 3 minutes, $z = 10^{10}$: protons and neutrons combine to form Deuterium (the isotope of Hydrogen with one proton and one neutron), which in turn converts almost immediately into Helium-4. The capture of neutrons to form nuclei prevents them from decaying further (the lifetime of the free neutron is about 15 min). The primordial abundance of atoms is determined in this process, which is known as big bang *nucleosynthesis* (BBN). For the lightest elements of the periodic table, the BBN abundance gets modified only marginally by subsequent astrophysical processes. The prediction of the abundance of light atoms is one of the greatest successes of the big bang theory.

- $T \sim 0.5$ keV, 2 months, $z = 2 \times 10^6$: the number of photons with energy of the order of T and above becomes effectively frozen because all active interactions (Compton scattering) conserve photon number. This is the *black-body time*, after which any process involving photons can destroy the black-body spectrum of the photons, which we eventually measure in the Cosmic Microwave Background radiation (CMB).

- $T \sim 1$eV, $z = 3300$: *matter-radiation equality*, where matter includes Dark (6 parts out of 7) and baryonic (1 part out of 7) matter and radiation is made of photons (60%) and neutrinos (40%). Dark Matter inhomogeneities start growing at this point to eventually form structures.

- $T \sim 0.3$ eV, around 370 ky, $z \simeq 1100$: *recombination* of electron and protons to form neutral Hydrogen (and earlier, at $z = 1400$ the recombination of Helium, which captures two electron per nucleus; this is a smaller effect and can be neglected for rough estimates). The fraction of free charged particles (e^- and p^+) decays very fast and very soon the photon cross section for Compton scattering becomes tiny. The universe becomes transparent. The photons travel freely (up to the 10% effect of re-ionization, see below) in every direction. It is these photons that we detect as CMB.

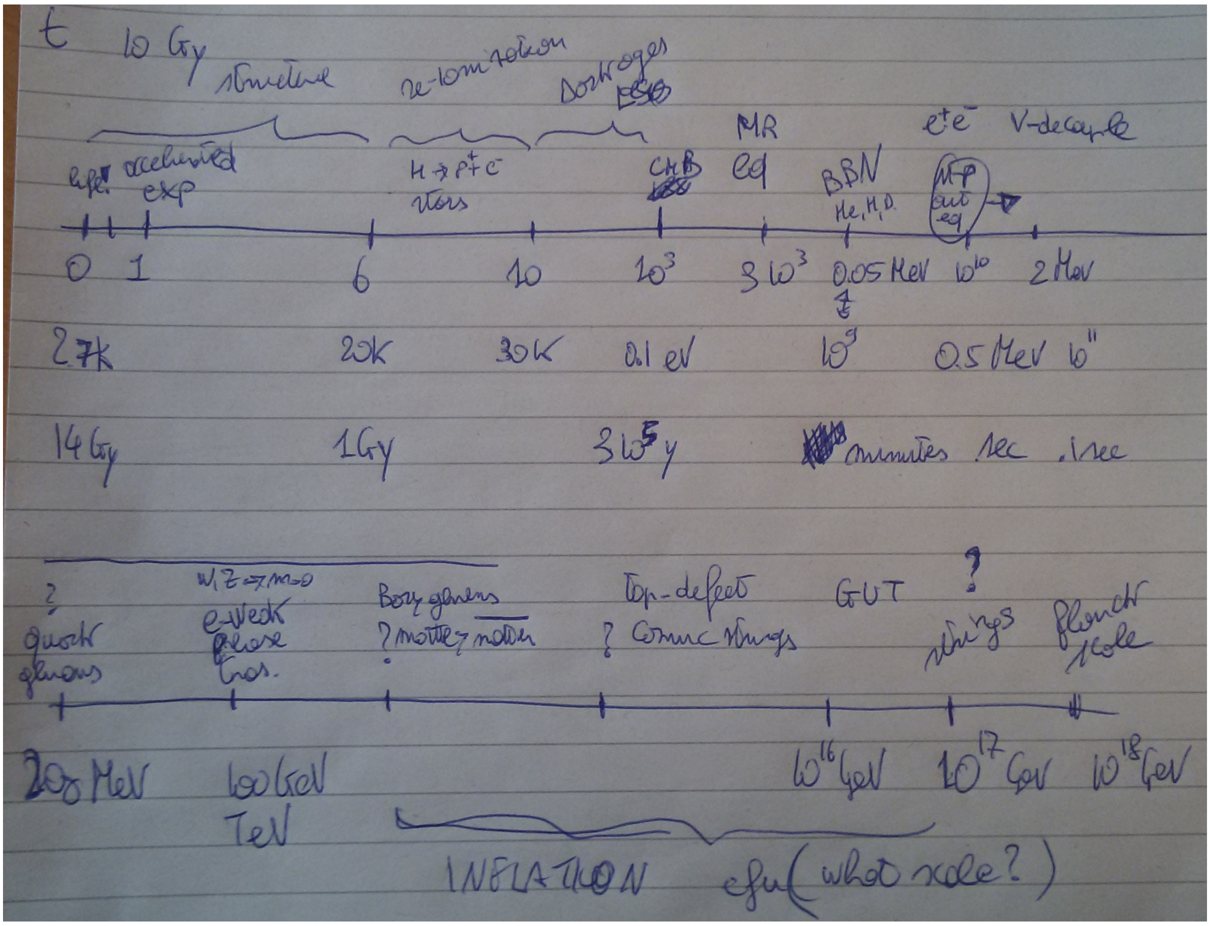


Figure 10:

- $z \sim 200$: radiation drag, i.e. baryonic matter, finally decouples from the photons (notice that there 10^9 photon every baryon, so photons decouple from baryons much earlier than baryons from photons) and starts falling into the gravitational potentials created by Dark Matter, which has been gravitationally clustering since matter-radiation equality.
- $z \sim 10$: most of the Hydrongen in the universe becomes ionized again as stars and galaxies become abundant. The detailed of this process, known as *reionization*, are still very uncertain and are expected to be clarified by ongoing and near future observations with large radio telescopes.
- $z \sim 0.3$, 9 Gy: the matter energy density equals that of Dark Energy and the universe enters a phase of accelerated expansion. Structure formation come to a stop because the expansion of the universe wins over gravitational collapse.
- $z \sim 0$, 14 Gy: these lecture notes are written.

6.3 Boltzmann equation

Our goal is to derive an equation to describe the evolution out of chemical equilibrium at various stages in the history of the universe. We will assume isotropy and homogeneity throughout. We will consider exclusively two-to-two body scattering, and use the notation $1 + 2 \leftrightarrow 3 + 4$ for the reaction of states or particles 1 through 4. We will assume that the reaction takes place in both directions. The variables we want to describe are the densities of species 1 though 4 as function of time in a flat FLRW universe. The *Boltzmann equation* for annihilation is given by[45]

[45] cfu: *Why the factor of $1/E_i$? To make the measure Lorentz invariant. It can be alternatively written as $\int d^3p_i dE_i\, \delta_D(E_i^2 - p_i^2 - m_i^2)$, with m_i the mass of the i-th particle*

z	Size	Temperature	Age	Comov. Dist.	Part. Horizon	Energy
0	14.3 Gpc	0.000234 eV	13.7 Gy	0.	14.3 Gpc	Λ
0.1	13.0 Gpc	0.000258 eV	12.4 Gy	414 Mpc	13.9 Gpc	Λ
0.39	10.3 Gpc	0.000326 eV	9.48 Gy	1.51 Gpc	12.8 Gpc	$\Lambda = \Omega_m$
1.	7.15 Gpc	0.000469 eV	5.92 Gy	3.32 Gpc	10.9 Gpc	Ω_m
3	3.57 Gpc	0.000937 eV	2.19 Gy	6.46 Gpc	7.81 Gpc	Ω_m
6	2.04 Gpc	0.00164 eV	947 My	8.42 Gpc	5.85 Gpc	Ω_m
10	1.3 Gpc	0.00258 eV	480 My	9.66 Gpc	4.61 Gpc	Ω_m
20	681 Mpc	0.00492 eV	181 My	11.0 Gpc	3.27 Gpc	Ω_m
50	280 Mpc	0.0120 eV	47.4 My	12.3 Gpc	2.00 Gpc	Ω_m
100	142 Mpc	0.0237 eV	16.8 My	12.9 Gpc	1.35Gpc	Ω_m
1100	13.0 Mpc	0.258 eV	369 ky	14.0 Gpc	280 Mpc	Ω_m
3200	4.47 Mpc	0.750 eV	56.9 ky	14.1 Gpc	119 Mpc	$\Omega_m = \Omega_r$
5×10^4	286 kpc	11.7 eV	292 y	14.3 Gpc	9.01 Mpc	Ω_r
$2. \times 10^6$	7.15 kpc	468.7 eV	68.0 days	14.3 Gpc	0.229 Mpc	Ω_r

Table 1: Numerical conversion among various measures of time.

$$a^{-3}\frac{d\left(a^3 n_1\right)}{dt} = \int \prod_{i=1,4} \frac{d^3 p_i}{(2\pi)^3 2E_i} \delta_D^3\left(\sum_{i=1,4} \vec{p}_i\right) \delta_D\left(\sum_{i=1,4} E_i\right) |M|^2 \tag{6.27}$$

$$\left[f_3 f_4 \left(f_1 \pm 1\right)\left(f_2 \pm 1\right) - f_1 f_2 \left(f_3 \pm 1\right)\left(f_4 \pm 1\right)\right] , \tag{6.28}$$

and similarly for particles 2 through 4. Several comments are in order:

- In the absence of any interaction, the right hand side vanishes and n_1 is covariantly conserved (i.e. it scales just with the volume)

$$0 = (u^\mu n)_{;\mu} = a^{-3}\frac{d\left(a^3 n_1\right)}{dt} \tag{6.29}$$

- $f_i(\mathbf{x}, \vec{p}, t)$ is the phase space density function. For today we will assume it does not depend on space.
- $(f_i \pm 1)$ come about because of the quantum statistic and are called Bose enhancement (easier to produce a boson in a state that is already occupied by a large number of particles) and Pauli blocking (one cannot have a density of state large than one for Fermions).
- the $f_3 f_4$ terms describes creation while the $f_1 f_2$ destruction of particle 1.
- the *probability amplitude* M gives the probability $|M|^2$ (quantum mechanics). This is the only place where the dynamics of the theory under consideration appears. M is proportional to the coupling constant responsible for the interaction.
- The delta functions ensure energy and momentum conservation in each interaction.
- the integrals over all four momenta sum over all possible ways that the interaction can proceed.
- This expression is *not* invariant under time reversal T. This is related to the (mysterious) fact that to a large extent (weak interactions being an exception that is nevertheless not sufficient to explain the mystery) microscopic physics is invariand under T, while all macroscopic process are observed to have an "arrow of time". In deriving this equation from the BBGKY hierarchy[46] we have neglected the 2-, 3-, ... n-particle densities and therefore we have lost the correlation among particles that is generated by interactions. This loss of information breaks T, even if the underlying interactions were T-symmetric, and lead to the possible increase of entropy and selects an arrow of time.

[46]If you don't know what this is an have not idea what I am talking about buty you are curious read chapter 68 of [?]

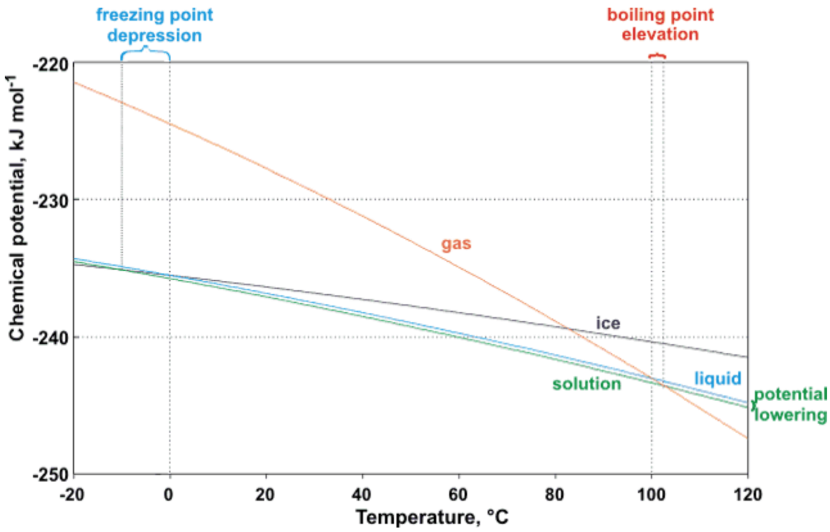


Figure 11: The chemical potential of different phases of water as function of temperature.

- This is a coupled system of non-linear, ordinary, integro-differential equations, a.k.a. it is pretty hard to solve

To make some progress we will make two simplifying assumptions:

1. There is *kinetic equilibrium*. This means that there are efficient interactions that distribute energy and momentum within a single species very quickly. It implies that we can use BE or FD statistic for the distribution functions f's (equilibrium distributions). Notice that this is *not* the same as *chemical equilibrium*, in which there are efficient interactions to change particles from one species to another. For example, in chemical equilibrium $\mu_1 + \mu_2 = \mu_3 + \mu_4$ (see below), but we will not assume this in the following. Intuitively this means that we consider a situation in which particles can change their energy and momentum but not necessarily their type.

2. In the three cases of interest we will have $T \ll E - \mu$ and therefore we can drop the ± 1 in the FD and BE statistic and simply use Boltzmann distribution functions ("classical")

$$f_{BE,FD} = \frac{1}{e^{(E-\mu)/T} \mp 1} \simeq e^{-(E-\mu)/T} = f_B \,. \tag{6.30}$$

   This also implies we can approximate $f_i \pm 1 \simeq 1$ since $e^{-(E-\mu)/T} \ll 1$

Let us define the species chemical equilibrium ($\mu = 0$) densities $n_i^{(0)}$ and out of equilibrium densities[47] n_i as (see Eq. (**??**))

$$n \equiv g \int \frac{d^3 p}{(2\pi)^3}\, e^{-(E-\mu)/T} = g\, e^{\mu/T} \begin{cases} \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} & T \ll m \\ \frac{T^3}{\pi^2} & T \gg m \end{cases} , \quad n^{(0)} \equiv n|_{\mu=0} = n\, e^{-\mu/T} \,, \tag{6.31}$$

with g the number of degrees of freedom (2 for the photon or the electron).

Assuming the chemical potential is momentum-independent, we can write

$$\left[f_3 f_4 \left(f_1 \pm 1\right)\left(f_2 \pm 1\right) - f_1 f_2 \left(f_3 \pm 1\right)\left(f_4 \pm 1\right)\right] \simeq e^{-(E_1+E_2)/T} \left[\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right] , \tag{6.32}$$

where we used the conservation of energy $E_1 + E_2 = E_3 + E_4$. Define the *thermally averaged cross section* as[48]

$$\langle \sigma v \rangle \equiv \frac{1}{n_1^{(0)} n_2^{(0)}} \int \prod_{i=1,4} \frac{d^3 p_i}{(2\pi)^3 2E_i} \delta_D^3 \left(\sum_{i=1,4} \vec{p}_i\right) \delta_D \left(\sum_{i=1,4} E_i\right) |M|^2 e^{-(E_1+E_2)/T} \,. \tag{6.33}$$

[47] cfu: *How can you compute these integrals by dimensional analysis?* $[n] = M^3$ *so for relativistic particles it must be* $n \propto T^3$.

[48] cfu: *Check dimensions:* $[\sigma] = L^2$, $[v] = L/T$, ...

We can finally write

$$\boxed{a^{-3}\frac{d\left(a^3 n_1\right)}{dt} = \langle\sigma v\rangle n_1^{(0)} n_2^{(0)}\left[\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right]}\,. \tag{6.34}$$

- Ordinary coupled differential equation
- Since $[n] = L^{-3}$ and $[\langle\sigma v\rangle] = L^3 T^{-1}$, we can think of $n_2\langle\sigma v\rangle$ as a reaction rate Γ ($[\Gamma] = T^{-1}$). Here, Γ is rate at which species 1 changes (created or destroyed) due to the reaction $1+2 \leftrightarrow 3+4$.
- The left hand side is of order n_1/t. When the right hand side is negligible, the variation of n_1 is determined exclusively by the expansion of the universe. Species 1 is diluted $\partial_t n_1 = -3Hn_1$.

Depending the rate of interaction and expansion of the universe, there are two relevant regimes

- *Equilibrium* $\Gamma \gg H$. he reaction is very efficient and determines the relative densities of species. Generic initial values for n_i are very quickly driven to the chemical equilibrium

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \qquad \Leftrightarrow \qquad \mu_1 + \mu_2 = \mu_3 + \mu_4\,, \tag{6.35}$$

  which ensures a large cancellation of the rhs. This is sometimes called Saha equation. Notice that, even if $\mu_i \neq 0$, the ratio of abundances is the same as it would be if $\mu_i = 0$ for every i, namely

$$\frac{n_3 n_4}{n_1 n_2} = \frac{n_3^{(0)} n_4^{(0)}}{n_1^{(0)} n_2^{(0)}}\,. \tag{6.36}$$

- *Freeze-out* $\Gamma \ll H$. The reaction is too slow to keep up with the expansion of the universe. One can neglect the right hand side of Eq. (6.34) and find $n_i(t) \simeq n_i(a_*)(a_*/a)^{-3}$, where $*$ refers to the (last) moment at which $\Gamma \simeq H$. Notice in particular that, after an interaction goes out of equilibrium the ratio of all species involved becomes constant (assuming there aren't other processes that affect them). This is called "freeze-out".

Recall that μ is akin to any other potential, e.g. the gravitational or electric potential. Imagine a single species. A larger number density implies a larger μ. If μ in region I is larger than μ in region II, then particles move from A to B. This is the macroscopic description of diffusion, e.g. of smoke in air. Now imagine a homogeneous system, so that μ is the same everywhere, but two species, A and B, which can freely transform into each other. If $\mu_A > \mu_B$ then all A particles will transform into B. This is what happens in a phase transition, e.g. when water becomes ice below $T = 0°C$ (see Fig. 11).

Problems for lesson 6

P.6.1 A useful relation during radiation domination is $T_{MeV} \simeq \mathcal{O}(1)\sqrt{t_{sec}}$. Derive it by yourself or following 1.

P.6.2 Reproduce table 1. Columns correspond to redshift z, Eq. (**??**), particle horizon d, Eq. (P.4.1), CMB temperature T, age of the universe $t_{\rm age}$, Eq. (4.25), comoving distance to a given redshift, Eq. (4.13), and type of energy domination.

P.6.3 Exercise 1 (D3) on the integrals of the phase space distribution

P.6.4 Exercise 6 (D3) on the baryon to photon ratio in our late universe.

LESSON 7

Out of equilibrium processes: Big Bang Nucleosynthesis ref

We use the Boltzmann equation to compute the out-of-equilibrium dynamics in three cosmological events: Big Bang Nucleosynthesis, recombination and Dark Matter decoupling.

7.1 Big Bang Nucleosynthesis

The baryons in the universe today are observed to be 75% Hydrogen (^{1}H) and 25% Helium (^{4}He) with only traces of the other isotopes and of the heavier elements (including us), see Fig. 13. Given that the typical binding energy of a nucleus is $\mathcal{O}(2-8)$ MeV per nucleon, at temperature $T \gg 2-8$ MeV, all the baryons in the universe were free protons and neutrons. At these temperatures, any atom would be instantaneously destroyed by some 8 MeV photon in the thermal bath. So it is natural to ask how the observed abundance of elements arose as the universe expanded and cooled much below this temperature[49]. This is the goal of *Big Bang Nucleosynthesis (BBN)*.

To study BBN analytically we will decompose the problem in two separated steps:

1. Calculation of neutron abundance, $T > 0.1$ Mev
2. Formation of Deuterium, Helium and heavier atoms, $T < 0.1$ Mev

This is a well justified separation for an estimate because the creation of atoms is heavily suppressed above 0.1 MeV.

Before we proceed, let us review the relevant species at MeV energies. Protons and neutrons are non-relativistic since $m_p \sim m_n \sim 1$ GeV$\gg$MeV and give a negligible contribution to the total energy density[50]. All neutrinos are relativistic and have just decoupled (around a few MeV, depending on the species). Electron and positron annihilate with each other quickly around $T \sim 0.5$MeV. This effect leads to a correction that is relevant when comparing to data, but which we will neglect in the following to keep the presentation simple.

Neutron abundance The abundance of neutrons and protons is related by the weak interactions. At MeV energies the effective Fermi theory contains the following two-body processes[51]

$$n + \nu_e \leftrightarrow p^+ + e^- \,, \quad n + \bar{e}^+ \leftrightarrow p^+ + \bar{\nu}_e \,, \quad n \to p^+ + e^- + \bar{\nu}_e \,. \tag{7.1}$$

We neglect for the moment neutron decay and come back to it later. If protons and neutrons remained in chemical equilibrium ($\mu_p = \mu_n = 0$), their ratio would be simply set by the usual Boltzmann suppression (use Eq. (6.31))

$$\frac{n_n}{n_p} = \frac{n_n^0}{n_p^0} = \frac{\left(\frac{m_n T}{2\pi}\right)^{3/2} e^{m_n/T}}{\left(\frac{m_p T}{2\pi}\right)^{3/2} e^{m_p/T}} \simeq e^{(m_n - m_p)/T} \equiv e^{Q/T} \,, \tag{7.2}$$

where the mass difference is $Q \equiv m_n - m_p \simeq 1.3$MeV and so $m_n/m_p \simeq 1.001$. So in equilibrium all neutrons would quickly decay away at $T < 1.3$ MeV. Luckily for us, the (weak) interactions responsible for this decay go out of equilibrium and some relic neutrons survive. This requires working out of chemical equilibrium, e.g. solving the differential Boltzmann equation Eq. (6.34).

Roughly we need temperature dependence of $\langle \sigma v \rangle$ and the $n_i^{(0)}$ and the time-temperature relation, so that we can solve for the ratio

$$X_n \equiv \frac{n_n}{n_n + n_p}. \tag{7.3}$$

Steps:

[49] It is straightforward to check (see P.7.1) that only a small fraction of He could have been synthesized in stars at later times.

[50] Recall that matter-radiation equality happens around $z_{eq} = \Omega_{m,0}/\Omega_{r,0} \sim 3500$, corresponding roughly to $T \sim$eV.

[51] Three and higher n-body processes are suppressed when the number densities n are low with respect to the typical interaction volume. In average, within an interaction volume d^3_{int} one finds nd^3_{int} particles. The probability for a given particle to interact at a given instant with a single other particle is then nd^3_{int}. The probability to interact with $k = 2, 3, \ldots$ other particles at the same instant is instead $(nd^3_{int})^k$. The latter possibility is very unlikely if $n \ll d^{-3}_{int}$.

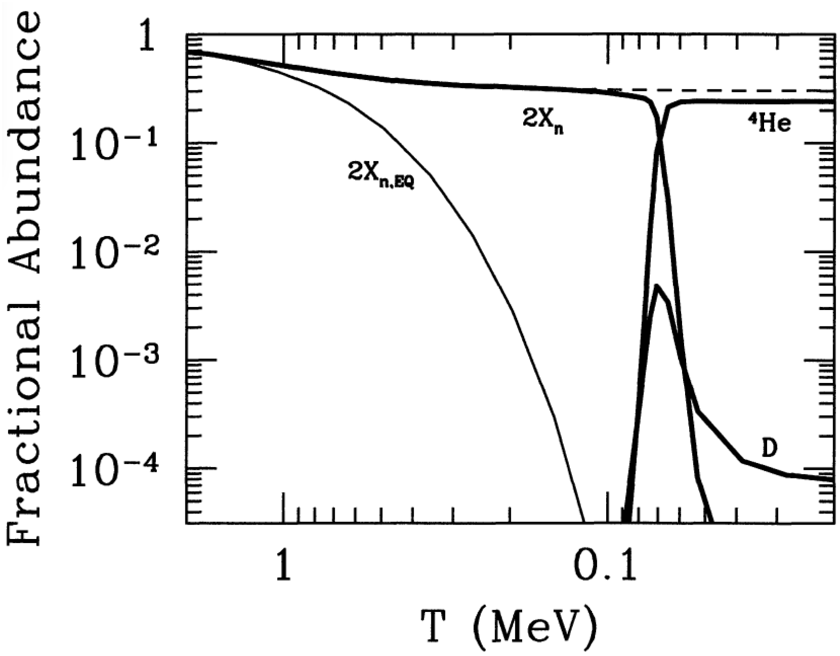


Figure 12: The fractional abundance of neutrons over baryons $(n+p)$ as function of temperature. Around $T \simeq 1$MeV neutrons go out of equilibrium and freeze out at 0.1MeV to $X_n \simeq 0.15$. BBN reactions become then relevant. Some neutrons decay and some combine to form Deuterium and Helium

1. We assume the leptons are in complete equilibrium: $n_l = n_l^{(0)}$. This means that the Boltzmann equation, Eq. (6.34), becomes
$$a^{-3}\frac{d\left(a^3 n_n\right)}{dt} = n_l^{(0)}\langle\sigma v\rangle\left\{\frac{n_p n_n^{(0)}}{n_p^{(0)}} - n_n\right\}. \tag{7.4}$$
2. Rewrite n_n in terms of the dimensionless X_n. We can use that, for all the weak processes in (7.1), the total number of baryons is conserved, so $(n_p + n_n)a^3$ is conserved. We get
$$\frac{dX_n}{dt} = \lambda_{np}\left[(1 - X_n)e^{-Q/T} - X_n\right], \tag{7.5}$$
where we introduce the neutron-proton conversion rate $\lambda_{np} = n_l^{(0)}\langle\sigma v\rangle$, which is time dependent.
3. Let us define a new dimensionless dependent variable $x = Q/T$ to substitute t. Use $\rho_r \propto a^{-4} \propto T^4$ to find
$$\frac{dx}{dt} = -x\frac{\dot{T}}{T} = xH \quad \Rightarrow \quad \frac{dX_n}{dx} = \frac{\lambda_{np}(x)}{xH(x)}\left[e^{-x} - X_n(1 + e^{-x})\right]. \tag{7.6}$$
To solve this we must make the x dependence of H and λ_{np}.
4. The calculation outlined in P.9.4 gives
$$\lambda_{np} = \frac{255}{t_{\text{life}}x^5}\left(12 + 6x + x^2\right), \tag{7.7}$$
with $t_{\text{life}} = 887$sec ~ 15 minutes.
5. To compute $H(x)$ we use the time-temperature relation during radiation domination (6.26), i.e. $T \propto t^{-1/2}$, with $g_* \simeq 10.75$ for photons $g_\gamma = 2$, three families of left-handed neutrinos and their anti-particle $g_\nu = 3 \times 2 = 6$, left and right-handed electrons and their anti-particles (positrons) $g_e = 2 \times 2 = 4$. This gives
$$H(x) = \frac{H(x=1)}{x^2} \simeq \frac{1.1\text{sec}^{-1}}{x^2} \tag{7.8}$$

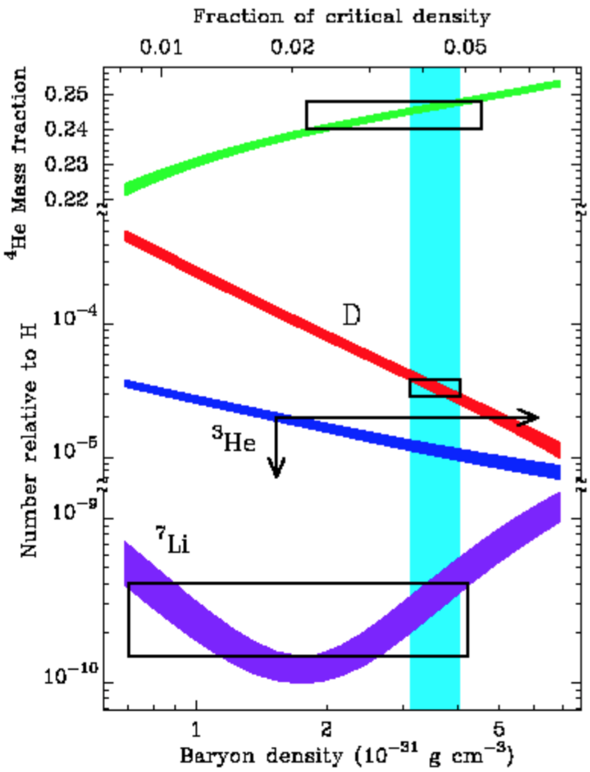


Figure 13: The abundance of the lightest elements (Deuterium, Helium 3 and 4, Lithium) as function of the baryon density today as predicted by BBN (colored bands). Black boxes represent the observational constraints. For $\Omega_b = 0.04$ all data are compatible with predictions.

6. Finally we solve (7.6) numerically. The result is plotted in Fig. 12. Compare $H(x = 1) \approx 1.13\,\text{sec}^{-1}$ and $\lambda_{np}(x = 1) \approx 5.5\,\text{sec}^{-1}$. After this, the collision rate drops much faster, so transition around 1 MeV.

7. After $T \sim 0.1$ MeV, neutron decay due to the process $n \to p + e^- + \bar{\nu}$ becomes important. This is easily taken into account by multiplying the number density of neutrons by $e^{-t/t_{\text{life}}}$ where $t_{\text{life}} \sim 15$ minutes is the neutron lifetime for the above process[52]. The decay reduces the neutron abundance by approximately 25 percent to $X_n(T_{nuc}) = 0.11$, before nucleosynthesis changes the story, which we discuss next.

Light element formation A good approximation to light element formation is that it happens instantaneously at some temperature T_{nuc} that can be calculated in equilibrium. The equilibrium abundance of Deuterium D is determined by the nuclear process

$$n + p \leftrightarrow D + \gamma\,. \tag{7.9}$$

Let us use the equilibrium condition, i.e. the vanishing of the right-hand side of the Boltzmann equation (6.34) adapted to the above process

$$\left[\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right] \quad \Rightarrow \quad \left[\frac{n_D n_\gamma}{n_D^{(0)} n_\gamma^{(0)}} - \frac{n_n n_p}{n_n^{(0)} n_p^{(0)}}\right] \tag{7.10}$$

Since photons have negligible chemical potential $n_\gamma = n_\gamma^{(0)}$, we find (see P.7.7)

$$\frac{n_D}{n_n n_p} = \frac{n_D^{(0)}}{n_n^{(0)} n_p^{(0)}}\,. \tag{7.11}$$

[52]cfu: *Why is the cosmological time t the right time to use? Because neutrons are non-relativistic $m_n \gg$Mev and so their proper time is well approximated by that of observers comoving with the Hubble flow, i.e. at constant comoving coordinates x. This is the definition of cosmic time t.*

Using again (6.31) or (6.17), this reduces to

$$\frac{n_D}{n_n n_p} = \frac{3}{4}\left(\frac{2\pi m_D}{m_n m_p T}\right)^{3/2} e^{(m_n+m_p-m_D)/T} \simeq \frac{3}{4}\left(\frac{4\pi}{m_p T}\right)^{3/2} e^{B_D/T}\,, \tag{7.12}$$

where we introduced the binding energy of Deuterium $B_D = m_n+m_p-m_D \simeq 2.2$ MeV and approximated $m_D \sim 2m_p \sim 2m_n$ in the fraction. We now drop order one factors, approximate $n_n \sim n_p \sim n_b$ and introduce the *baryon-to-photon* ratio [Problem P.6.4]

$$\eta_b \equiv \frac{n_b}{n_\gamma} = \frac{\rho_b}{m_p}\frac{\pi^2}{2\zeta(3)T^3_{\rm CMB}} \simeq 5\times 10^{-10}\,. \tag{7.13}$$

Finally, (7.12) becomes

$$\frac{n_D}{n_b} \simeq \eta_b \left(\frac{T}{m_p}\right)^{3/2} e^{B_D/T}\,, \tag{7.14}$$

The physics behind this equation is that the process $D+\gamma \leftrightarrow p+n$ happens in both ways as long as there are enough photons with energy of order 2.2 MeV, which are able to break up Deuterium. Naively one would expect this to stop being true at $T_{\rm naive} \sim 2.2$MeV. This is too rough though, because there are a billion photons per baryon Eq. (7.13). Even when $T < 2.2$MeV, there are still enough photons in the hot tails of the phase space distribution to destroy Deuterium. Deuterium remains in equilibrium well past 2.2 MeV. Solving for the $T=T_D$ when $n_D \simeq n_b$, one finds[53]

$$T_D \gtrsim \frac{2.2}{\log(5\times 10^{-10})}\,\text{MeV} \sim \frac{2.2}{20}\text{MeV} \simeq 0.1\text{MeV}\,. \tag{7.15}$$

This means that we assume that around 0.1 MeV roughly all remaining neutrons combine with protons to form deuterium. Actually, since the binding energy for Helium is higher than that of deuterium, very soon after T_{nuc}, the helium abundance grows larger than the deuterium abundance. It is therefore a good approximation to assume all neutrons go into ^4He. Since two neutrons go into one helium nucleus, our prediction for the helium mass fraction of the total amount of baryons is

$$X_4 \equiv \frac{4n_{^4He}}{n_b} \approx 2X_n(T_{nuc}) = 0.22. \tag{7.16}$$

Good approximation to actual result: one of the pillars of observational cosmology!
Note: deuterium doesn't completely disappear. Freeze out turns out to be very sensitive to η_b: great probe! Find $\Omega_b h^2 = 0.0205 + -0.0018$, which is a very good CMB-independent probe. Show plot.

7.2 Recombination*

Process:

$$e^- + p^+ \leftrightarrow H^0 + \gamma. \tag{7.17}$$

Happens around ~ 1 eV. Again compare with hydrogen binding energy of 13.6 eV. We are going to track the electron to hydrogen abundance (so the effect of the expansion of the universe drops out)

$$X_e \equiv \frac{n_e}{n_e+n_H} = \frac{n_p}{n_p+n_H}, \tag{7.18}$$

where the latter equality is ensured by the neutrality of the universe. Note that this is in principle not the quantity that measures the kinetic equilibrium of the photons: the photons are chemically decoupled from the fluid since 1 MeV, but remain in kinetic equilibrium until decoupling of photons from matter, which we investigate later. Due to the imbalance between photons and baryons, the decoupling of matter from photons is yet another question. Our discussion of recombination will be very similar to the neutron-to-proton ratio story. Once again, one can start from equilibrium considerations to find the temperature at which the ratio starts to change significantly. The equilibrium condition (Saha equation) is again

$$\frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}\,, \tag{7.19}$$

[53] cfu: *What is the interpretation of the other large factor* $(3/2)\log(T/m_p) \simeq -14$*?*

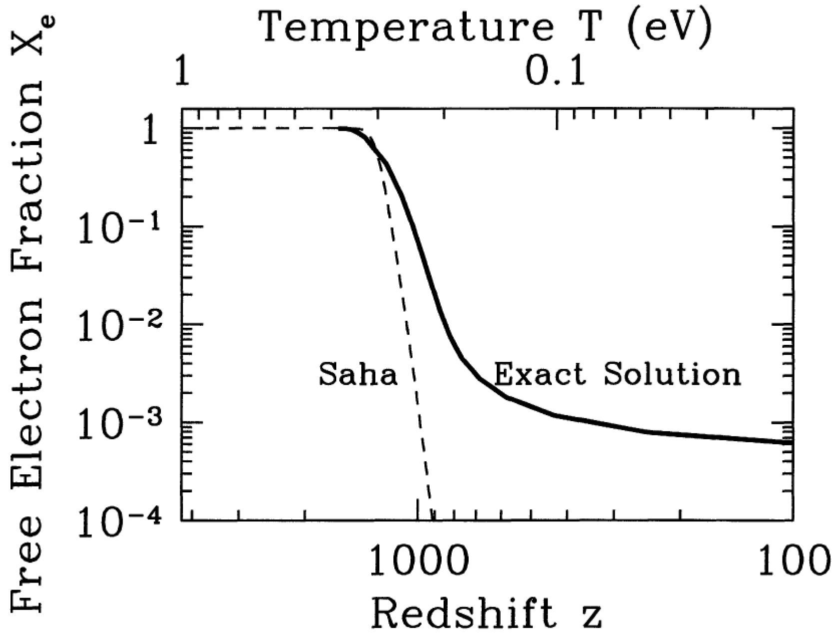


Figure 14:

which can be written as

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_e + n_H}\left[\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T}\right], \tag{7.20}$$

with $\epsilon_0 = m_e + m_p - m_H$. At $T \sim 13.6$ eV, this gives us a tiny amount of neutral H. At $T \sim 1$ eV, or $z = 1000$ neutral H grows, but the process goes out of equilibrium and the full differential equation Eq. (P.9.4) needs to be solve.

The equation governing the electron fraction going out of equilibrium is

$$\frac{dX_e}{dt} = \langle\sigma v\rangle \left\{(1-X_e)\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T} - X_e^2 n_b\right\}. \tag{7.21}$$

Very similar to previous case. Difference is in the fact that electron mass matters and $n_e = n_p$ (explains the square appearing) and we use $n_e + n_H = n_b$. For the cross section we need

$$\langle\sigma v\rangle = \alpha^{(2)} = \frac{10\alpha^2}{m_e^2}\left(\frac{\epsilon_0}{T}\right)^{1/2} \ln\left(\frac{\epsilon_0}{T}\right). \tag{7.22}$$

Draw energy levels of hydrogen.
[54]

Fig. 14 show the result of the numerical integration.

Decoupling Photons remain in kinetic eq. mainly due to Thomson scattering off electrons, $\sigma_T \simeq 0.7 \times 10^{-24}$ cm^{-2}. They go out of equilibrium when this rate becomes comparable to the expansion rate of the universe:

$$n_e \sigma_T = X_e n_b \sigma_T \sim H^{-1}. \tag{7.23}$$

Plugging in the numbers we find [Problem P.7.8]

$$\frac{n_e \sigma_T}{H} = 113 X_e \left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{0.15}{\Omega_m h^2}\right)^{1/2}\left(\frac{1+z}{1000}\right)^{3/2}\left[1 + \frac{1+z}{3600}\frac{0.15}{\Omega_m h^2}\right]. \tag{7.24}$$

[54] cfu: *Why $\alpha^{(2)}$? Answer: 13.6 eV photon is reionizing. Formation through cascade.*

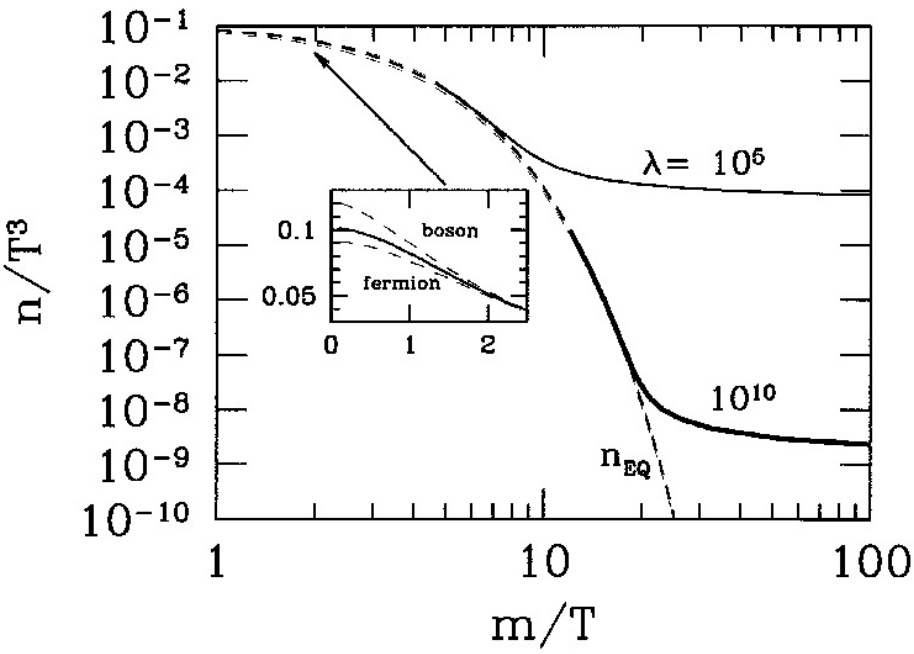


Figure 15:

So around $z = 1100$, the photons decouple when the electron fraction becomes of order 10^{-2}. We see from the numerical solution this indeed quickly happens around $z = 1100$, so photons decouple (CMB emission) at the time of recombination. Note that even if for some spurious reason all hydrogen gets reionized at some point, the photons will still decouple later, solely due to the expansion of the universe. This is relevant since we actually think such a reionization event took place before $z = 6$. Solving the rate equality for $X_e = 1$ tells us that photons decouple around $z = 43$ regardless. This is why only around 10 percent of CMB photons rescatter at this reionization event and the primordial data are not distorted too much. The latest PLANCK results estimate reionization took place instantaneously at $z = 8.8^{+1.7}_{-1.4}$.

7.3 Dark matter decoupling*

We investigate the WIMP scenario here. There are other scenario's, such as decaying DM, which allow one to search for DM masses in a wide range. The WIMP scenario however predicts GeV masses. Process:

$$X + X \leftrightarrow l + l, \tag{7.25}$$

where X is the heavy DM particle and l is a light known particle that DM weakly interacts with. The light particles are in complete chemical as well as kinetic eq. The equation governing the DM fraction

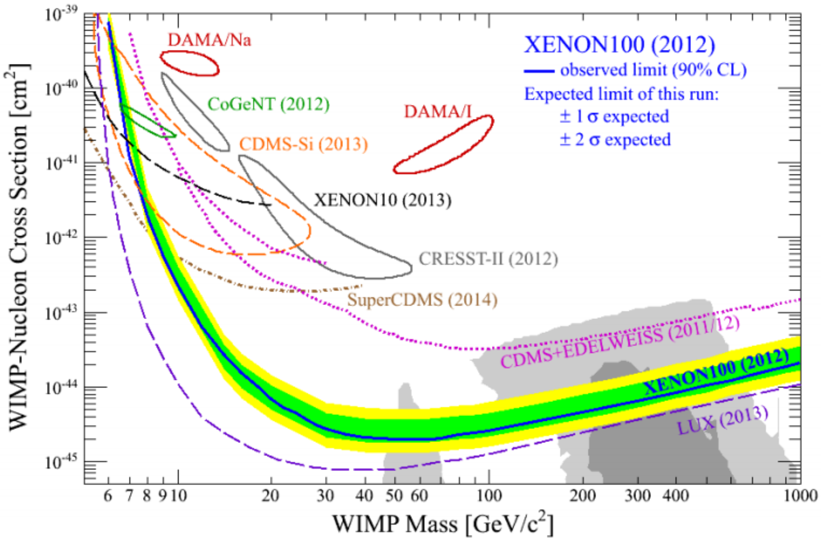
$$Y \equiv \frac{n_X}{T^3}, \tag{7.26}$$

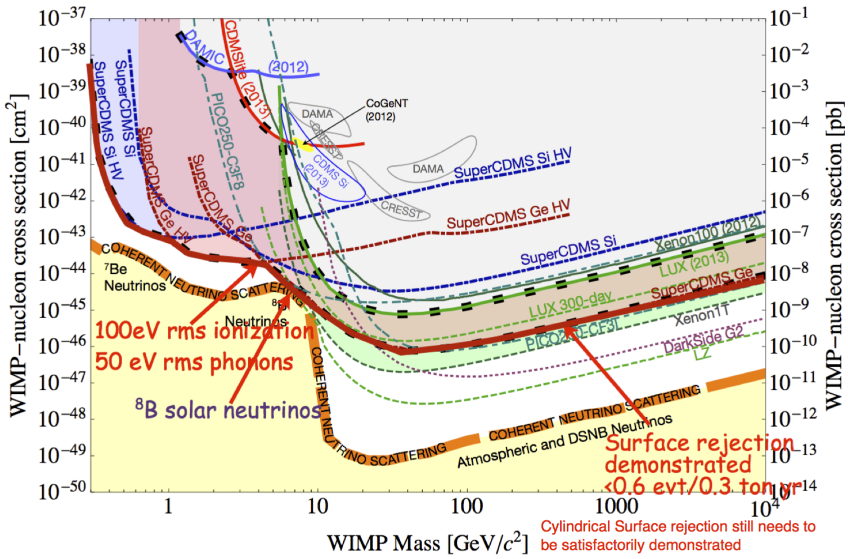
now becomes

$$\frac{dY}{dt} = T^3 \langle \sigma v \rangle \left\{ Y_{EQ}^2 - Y^2 \right\}, \tag{7.27}$$

with $Y_{Eq} \equiv n_X^{(0)}/T^3$. At very high temperatures $T \gg m$, Dark Matter was relativistic and $Y_{Eq} = 1$. Note that when the process goes out of equilibrium, $Y > Y_{EQ}$, and Y decreases with time, so the sign in this equation is correct. The T^3 basically comes from the fact that the relevant era here is radiation, during which $a \sim T^{-1}$. Again we would need to know the temperature dependence and size of the cross section when the temperature is of the order of the DM particle mass. (Comment on approximation for and definition of λ?) Let us use $x \equiv m/T$ as time, then we find the master equation

$$\frac{dY}{dx} = -\frac{\lambda}{x^2} \left[Y^2 - Y_{Eq}^2 \right], \tag{7.28}$$





with $\lambda \equiv m^3\langle\sigma v\rangle/H(m)$. After looking at the numerical solution in Fig. 15, we can derive a rough estimate of the freeze out abundance

$$\frac{dY}{dx} \sim -\frac{\lambda}{x^2}Y^2 \quad \Rightarrow \quad Y_\infty \sim \frac{x_f}{\lambda} \sim \frac{10}{\lambda}\,. \tag{7.29}$$

After freeze-out, the energy density falls off as $1/a^3$. However, just like for neutrinos, the photon fluid temperature develops slightly differently due to the decoupling of all massive particles in the range 100 GeV till now. Therefore the DM energy density today is

$$\rho_X = mY_\infty T_0^3\left(\frac{a_1T_1}{a_0T_0}\right)^3 \approx mY^\infty T_0^3/30. \tag{7.30}$$

where $(a_1T_1)/(a_0T_0) \sim 1/30$ arises from the fact the number of degrees of freedom around $T \sim GeV$ was about 100, while it is a few today [Problem P.7.4] Then one can make prediction of Ω_X: insensitive to DM mass, since $Y_\infty \sim x_f \sim 1/m$: energy density does not depend on mass (apart from indirect dependence of $g_\star$ and the final ratio m/T_f). WIMP mass high compared to SM particles. Then we can estimate the relevant cross sections: a few orders of magnitude below estimates from supersymmetry. Show plots.

Problems for lesson 7

P.7.1 Compute the luminosity-to-mass ratio that stars would need in order to synthesize the observed 25% of ^{4}He during the last 14 billion years. Compare the result to the observed luminosity-to-mass (baryonic) ratio observed in the universe, $L/M_b \lesssim 0.05 L_{\odot}/M_{\odot}$, where the label $\odot$ refer to our sun. [Hint: compute the energy per baryon from the binding energy of He. Divide by the age of the universe and compare with $L_{\odot}/M_{\odot}$]

P.7.2 (From Dodelson Ch 3, Ex 2) Track the density of electron and positron during BBN. Since electromagnetic interactions are very strong during BBN, you can estimate this using $\mu_{e^-} = \mu_{e^+} = \mu_\gamma = 0$. When does the energy density of n_e fall to 1% of that of photons?

P.7.3 (From Dodelson Ch 3, Ex 6) Determine the baryon to photon ratio and show it is approximately given by

$$\eta_b \equiv \frac{n_b}{n_\gamma} = 5 \times 10^{-10} \left(\frac{\Omega_b h^2}{0.02} \right) \tag{7.31}$$

P.7.4 (from Dodelson Ch 3, Ex 11) As long as g_* is constant, the conservation of total entropy $s \propto a^{-3}$ plus the relation $s \propto g_* T^3$ (since entropy is dominated by relativistic species) implies $T \propto 1/a$. Compute aT at $T = 10$ GeV and today to quantify how much our universe deviates from the simple inverse scaling relation, due to the change in g_*.

P.7.5 Exercises 7 (D3) on the baryon loading for recombination (don't forget neutrinos)

P.7.6 *Optional* Exercise 12 (D3) on the density of baryon in the absence of baryogenesis

P.7.7 *Optional* Estimate the Deuterium abundance assuming chemical equilibrium Eq. (7.15). [Hint: use the chemical equilibrium condition Eq. (6.35) and the binding energy of Deuterium $m_p + m_n - m_D \simeq$ 2.2 MeV.]

P.7.8 Derive equation Eq. (7.24)

Check for understanding of lesson 7

cfu.7.1 Why does BBN take place at $T \sim 0.1$ MeV, which is about 20 times colder than the binding energy or Deuterium, $B_D \sim 2.2$ MeV?

cfu.7.2 How much has the universe expanded since the end of inflation until today in term of $N \equiv \ln a_i/a_0$? Assume (as it is the case) that the scale of inflation is completely unknown and that we do not want to spoil the successes of BBN. What is the lowest possible value of N allowed by observation?

cfu.7.3 How does BBN constrain the abundance of additional light degrees of freedom beyond the standard model that interact only gravitationally, including e.g. a stochastic background of gravitational waves?

LESSON 8

Inflation: Motivations

ref

In this section I discuss several problems with any cosmological model in which the universe is dominated by radiation in the far past, all the until the Big Bang. I will refer to this class of models collectively as "Hot Big Bang" model, where "hot" refers to the temperature of radiation. In particular, the root of all problems will be that most[55] of the expansion ($\dot{a} > 0$) of the universe e.g. in ΛCDM is decelerated $\ddot{a} < 0$. Decelerated expansion starts from the Big Bang (which in ΛCDM would happen during radiation domination) at $z \to \infty$ or $a \to 0$ and lasts all the way until Dark Energy takes over "recently" around $z \simeq 0.5$. First, I discuss old "background" problems, namely the horizon and curvature problems, which can be stated already for the unperturbed FLRW universe that we have studied so far. These problems were originally formulated in the 80's and have not changed much since. Second, I discuss new "perturbation" problems, namely scale invariance and phase-coherence problems, which have to do with the large amount of new data we have collected in the past 30 years, especially from the Cosmic Microwave Background (CMB). Finally, in preparation for the next lecture, I review the basic properties of the maximally symmetric spacetime with positive cosmological constant, i.e. de Sitter spacetime

8.1 Old background problems

In the following I discuss two of the problems that were well known more than 40 years ago and pushed many cosmologists to modify the early expansion history of our universe.

8.1.1 Curvature problem

The first background problem is that we do not observe any spatial curvature in our universe, despite the fact that curvature dilutes more slowly than radiation and matter (and in fact than anything obeying the SEC) and show grow with time relatively to them. Let us see this in formulae.

Current bounds tell us that [1]

$$\Omega_K \equiv \left(\frac{K}{a^2 H^2}\right) , \quad \Omega_0 = 0.000 \pm 0.005 \,. \tag{8.1}$$

On the other hand, as we saw in Lecture 3, the most general homogeneous and isotropic space can have spatial curvature, i.e. $K \neq 0$. From Eq. (8.1) we see that Ω_K grows with time in an decelerated ($\ddot{a} < 0$) expanding ($\dot{a} > 0$) universe

$$\dot{\Omega}_K = -\ddot{a}\frac{2K}{\dot{a}^3} \propto -\ddot{a} \propto (\rho + 3p) \propto (1 + 3w) \,, \tag{8.2}$$

where in the second step I used the acceleration equation (3.52) to show that in an expanding universe ($\dot{a} > 0$) the Strong Energy Condition (SEC, see (5.16)) implies deceleration. Since at early times in ΛCDM the universe is dominated by radiation, $w = 1/3$, we conclude that Ω_K must have been even smaller in the past[56]. In other words, extrapolating closer and closer to the Big Bang singularity at $a \to$ and $\rho \to \infty$, we are forced to assume that the initial curvature was tiny, $\Omega_K(a_i) \to 0$, or equivalently the initial total density of the universe was extremely close to the critical one, $\sum_i \rho_i \to \rho_c$ (defined in 3.46). There are only three logical possibilities:

1. The curvature of the universe is zero to begin with, and so it did not grow with time. While this is a possibility in an exactly homogeneous universe, it is very unlikely be realized in our universe because we observe non-vanishing perturbations on all scales. In particular, we measure deviations from exact FLRW of order $\Delta(\lambda) \sim 10^{-5}$ at wavelength λ of order the (physical) Hubble radius $\lambda \sim 1/H$. These perturbations are approximately scale invariant for shorter scales, $\lambda < 1/H$ and so it is natural to expect that there exists non-vanishing perturbation of a similar amplitude on superHubble scales $\lambda \gtrsim 1/H$. Such perturbations would induce a local spatial curvature of the order

$$K = \frac{\Delta(\lambda)}{\lambda^2} \quad \Rightarrow \quad \Omega_{K,0} = \frac{\Delta(\lambda)}{\lambda^2 H_0^2} \lesssim 10^{-5} \,. \tag{8.3}$$

[55] This is measured on a log scale, i.e. the duration of a cosmological phase is measured in terms of $\log\left(a_f/a_i\right)$, where $a_{i,f}$ are the initial and final value of the scale factor.

[56] CFU: *Estimate Ω_K at Big Bang Nucleosynthesis.*

This argument strongly disfavours this possibility.

2. The initial conditions of the universe, as it emerged from some yet unknown non-perturbative theory of quantum gravity[57], were extremely fined tuned close to Ω_K. In this scenario, the existence of the universe as we know it is a very rare fluctuation, since any larger initial value of $\Omega_K(t_i)$ would have grown to dominate the energy density of the universe and prevented the formation of galaxies and therefore life as we know it. Also not a great option, in the opinion of many.

3. The early expansion history of our universe is modified to stop Ω_K from growing as we move back in time. From (8.2) we see that this requires either $\ddot{a}, \dot{a} < 0$, i.e. an early phase of decelerated contraction, or $\ddot{a}, \dot{a} > 0$, i.e. an early phase of accelerated expansion. Since we know the current universe is expanding (recall Hubble's law), the first of these options requires to *bounce* i.e. to transition from $\dot{a} \propto H < 0$ to $\dot{a} \propto H > 0$. Achieving the bounce in a controlled construct is still an open problem and the many proposed models have a series of pathologies, as discussed in Box (1). Therefore we focus an early phase of accelerated expansion, a.k.a. cosmological *inflation*, in the rest of these notes.

Summarising, to avoid fine tuned initial conditions for the universe, we postulate the existence of a primordial phase of accelerated expansion, $\ddot{a}, \dot{a} > 0$, called inflation.

Box 8.1 Bouncing Universes To transition from a contracting phase, $H < 0$ to an expanding one, $H > 0$, we need $\dot{H} = 0$. ...

8.1.2 Horizon problem

A second background problem of the Hot Big Bang model is that the homogeneity of the observed universe on large scales is at odds with the decelerated expansion history. In fact, cosmological observations of far away objects allow us to see regions in the past that are much larger than the particle horizon at the time. Any mechanism attempting to explain the observed homogeneity in a causal way then necessarily violates causality, leading the horizon problem.

To see this quantitatively, recall that the comoving distance (see Lecture 4) between two generic times t_1 and t_2 with $a_1 = a(t_1) < a(t_2) = a_2$ is found to be

$$\chi(a_1, a_2) \equiv \int_{a_1}^{a_2} \frac{da}{a^2 H} = \frac{1}{a_1 H_1} \frac{2}{3w+1} \left[\left(\frac{a_2}{a_1} \right)^{(3w+1)/2} - 1 \right], \tag{8.4}$$

where I assumed $w \neq -1/3$. Then, the distance of an object at redshift $1+z = a^{-1}$ from us at $a = a_0 = 1$ is given by

$$\chi(a, 1) \equiv \int_{a}^{a_0} \frac{d \log a}{aH} = \frac{1}{H_0} \frac{2}{3w+1} \left[1 - a^{(3w+1)/2} \right], \tag{8.5}$$

Imagine now to look out in the night sky in opposite directions and detect a pair of antipodal object, each sending us radiation with the same[58] redshift z. The relative comoving distance $\Delta\chi$ between the objects is just $2\chi(a, 1)$. To simplify the algebra, let us neglect Dark Energy[59] and so $w > -1/3$ (In ΛCDM $w \in \{0, 1/3\}$) and assume $a \ll 1$. Then

$$\Delta\chi(a, 1) \simeq 2 \times \frac{1}{H_0} \frac{2}{3w+1} \simeq \frac{\mathcal{O}(1)}{H_0}, \tag{8.6}$$

Recall that the redshift of these objects is $1 + z = 1/a$, and so we conclude that high redshift objects $z \gg 1$ are at a distance of order the Hubble radius today H_0^{-1}, almost independently of z.[60] Since this is

[57] cfu: *Strictly within GR, K is just a parameter, not a dynamical variable, and so there in no physical perturbation that can make $\Omega_K = 0$ unstable. On the other hand, GR is most likely just a low-energy (subPlanckian) effective description of some UV-complete theory of quantum gravity, and it is at least plausible that $\Omega_K = 0$ might be unstable within that larger, yet unknown theory. Perhaps a more concrete example is bubble nulceation. instanton solutions are known in which a new universe nucleates from a single point []. To respect the isometries of the system the new universe must have some negative curvature. It is not known whether bubble nucleation and the ensuing ideas about the multiverse play a role in the history of our own universe, and the discussion among experts continues.*

[58] This assumption is clearly not necessary, but it allows us to avoid obfuscating ideas with indices.

[59] cfu: *Check that this does not affect the argument at all.*

[60] CFU: *Using the Hubble law, show that the Hubble radius H^{-1} represents the physical distance beyond which comoving object move away from us faster than the speed of light, namely $\partial_t x_{phy} > c = 1$.*

a comoving distance between objects at fixed comoving position (i.e. far away object are in the Hubble flow), it does not depend on time. Let us compare now this distance with the comoving particle horizon in a hot Big Bang model, i.e. extrapolating radiation domination all the way to $a_i = 0$. Recall that the comoving particle horizon[61] $x_{\text{p.h.}}$ is the comoving distance traveled by light since the beginning of time τ_i, namely $x_{\text{p.h.}}(a) \equiv \chi(a_i, a)$. Notice that $x_{\text{p.h.}}$ depends on the integral in (8.5) over the whole history of the universe, as opposed for example to the Hubble radius r_H, which carries information about a single instant of time. Recall also that for $w > -1/3$, or equivalently decelerated expansion $\ddot{a} < 0$ (as it is the case for radiant and dust), one can safely take $a_i \to 0$ and so $x_{\text{p.h.}}(a)$ equals the comoving Hubble radius[62] times an order one number[63]

$$x_{\text{p.h.}}(a) = \frac{1}{aH}\frac{2}{3w+1} \simeq \frac{1}{aH}\mathcal{O}(1) \simeq r_H(a)\mathcal{O}(1) \quad (\text{decelerated})\,. \tag{8.8}$$

Assuming decelerated expansion since the Big Bang, one finds

$$\frac{\Delta\chi(a)}{x_{\text{p.h.}}(a)} \simeq 2\,\frac{aH}{a_0H_0} \simeq 2\left(\frac{1}{a}\right)^{(3w+1)/2} \gg 1 \quad (\text{decelerated})\,. \tag{8.9}$$

We just learnt that, in an ever decelerating universe, by observing far away objects ($1/a = 1+z \gg 1$) we are actually probing scales much larger than the particle horizon at that time. In practice, one can reach $a = (1+z)^{-1} \sim 0.1$ with quasar and $a \sim z^{-1} \sim 10^{-3}$ with Cosmic Microwave Background (CMB) photons. In both cases, the observed physical properties (e.g. density of quasars, temperature and polarization of the CMB) are the same in the opposite directions in average. We conclude that, in the absence of accelerated expansion in our past, the mechanism responsible for this observed statistical isotropy must violate causality. This is the *particle horizon problem.*

Conversely, for a phase of accelerated expansion, $\ddot{a} > 0$ or $w < -1/3$ (such as during Dark Energy or inflation) during a period $a \in \{a_i, a_f\}$, the result is divergent as $a_i \to 0$:

$$\begin{aligned} x_{\text{p.h.}}(a_f) &= \frac{1}{a_fH_f}\frac{2}{|3w+1|}\left[\left(\frac{a_f}{a_i}\right)^{|3w+1|/2} - 1\right] && (8.10)\\ &\simeq \frac{1}{a_fH_f}\frac{2}{|3w+1|}\left(\frac{a_f}{a_i}\right)^{|3w+1|/2} \gg r_H \quad (\text{accelerated})\,. && (8.11)\end{aligned}$$

In the extreme case $w \simeq -1$ (inflation), H is approximately constant and $x_{\text{p.h.}}$ asymptotes to the constant value

$$x_{\text{p.h.}} \to \frac{1}{a_iH_i} \quad (\text{inflation})\,. \tag{8.12}$$

Yet again, if we want to keep causality as a guiding principle, we must postulate a phase of accelerate expansion $\ddot{a}, \dot{a} > 0$ in the early universe[64] (or a phase of decelerated contraction $\ddot{a}, \dot{a} < 0$, with the problems discussed in Box 1).

The horizon problem is well summarised by the plot in Fig. **??**, which shows the time evolution of x_H and $x_{\text{p.h.}}$ for our universe. The ordinate represents time and is parameterized by the number of *e-foldings* of expansion

$$dN \equiv Hdt = d\log a \quad \Rightarrow \quad N = \log a + \text{const}\,. \tag{8.14}$$

[61] This is simply related to the physical particle horizon $d_{\text{p.h.}}$ of (4.18) by $ax_{\text{p.h.}}(a) \equiv d_{\text{p.h.}}(a)$

[62] Recall the comoving *Hubble radius* r_H, which is defined as

$$r_H \equiv \frac{1}{aH} = \frac{1}{\dot{a}} = \frac{a^{(3w+1)/2}}{H_0} \qquad (\text{single fluid})\,, \tag{8.7}$$

where in the third equality I used the solution of the Friedmann equation for a single fluid with $p = w\rho$ and constant w. As usual, the physical Hubble radius is simply $r_{H,\text{phys}} = ar_H = H^{-1}$. In the literature, r_H is often referred to as *Hubble "horizon"*. This is a misnomer since neither $(aH)^{-1}$ nor its physical cousin H^{-1} represent a horizon in the usual sense of GR. This nomenclature is widely spread and not harmful as long as one is aware of the subtleties. In these notes, I will try hard to use the expressions"Hubble radius" or just "Hubble scale" instead of "Hubble horizon".

[63] CFU: *Show that if two different decelerated phase follow each other (radiant and matter domination in our universe), the contribution from the latter dwarfs that of the former.*

[64] CFU: *Prove that, during decelerated expansion, $\ddot{a} < 0$, perturbations "re-enter" the Hubble horizon, in the sense that*

$$\frac{\partial}{\partial t}\left(\frac{\lambda_{phy}}{H^{-1}}\right) < 0\,, \tag{8.13}$$

and viceversa.

Figure 16: The plot shows the evolution of the comoving distance and the particle horizon in the phase of early accelerated and late decelerated expansion. The large growth of the particle horizon during inflation ensures that it is causally possible for any point in the current observable universe today to exchange information with any other.

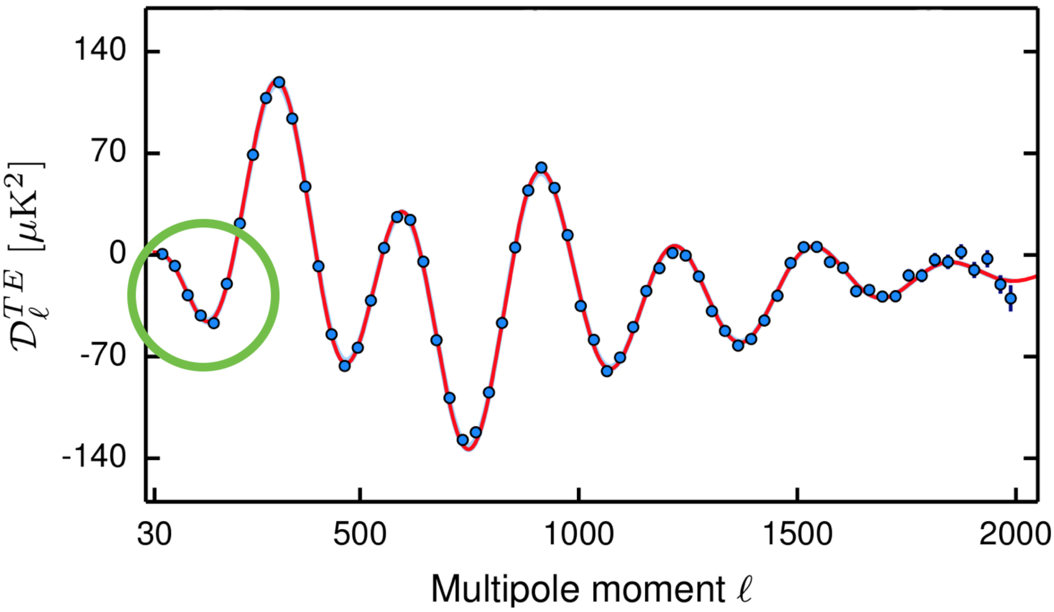


Figure 17: The plot shows the cross-correlation between CMB temperature T and E mode polarization [1]. The *anti*-correlation around $l \sim 100$ shows that superHubble perturbations at the time of last scattering exist and they oscillate with coherent phases.

I have chosen the integration constant so that $N = 0$ separates early accelerated expansion, i.e. inflation, from late deceleration, i.e. radiation and matter domination. The abscissa of the upper and lower panels indicates physical and comoving scales respectively. The black lines represent Hubble radius, while ... Diagonal, thin, red lines represent the physical wavelength λ_{phy} or comoving wavelength λ of some (monocromatic) perturbation.

Box 8.2 Topological defects To be written

8.2 New perturbation problems

There are problems with the hot Bib Bang models that were not know 40 years ago because the data was not good enough. I believe these “new” problems must play an important role in guiding us towards a theory of the early universe.

8.2.1 Phase coherence problem

As we saw discussing the horizon problem, by observing distant objects at $z \gg 1$, we can see scales much larger than the Hubble radius at the time. Our universe does have perturbations already on these superHubble scales, i.e. with wavelength $\lambda > 1/H$. What's really remarkable is that all these superHubble perturbations we have observed appear to oscillate in exact synchronicity: they have all the same phase! This is the *phase coherence* of cosmological perturbations, which give rise to the distribution of galaxies in the universe today. In an ever decelerating universe, the Hubble radius and the particle horizon are the same up to an unimportant order one factor. In this case then phase coherence is observed even on scales much large than the particle horizon. This is a problem because on these super-horizon scales no causal mechanism can be devised to “synchronized” the phases and so their coherence becomes a very unlikely coincidence. This strongly suggests that there was a primordial phase, before the hot

Big Bang, during which perturbations were produced and synchronized, rather than being generated at "late" time, during the hot Big Bang. Let us see this more in detail.

In the CMB, for each direction $\hat{n}$ of the sky ($\hat{n}\cdot\hat{n}=1$), we observe both temperature fluctuations $\Delta T(\hat{n})\equiv T(\hat{n})-\bar{T}$ around the average temperature $\bar{T}$, and a specific type of photon polarization called E-mode and denoted by $E(\hat{n})$. Because of the isotropy of the universe on large scales, it is convenient to decompose fields on the sphere into spherical harmonics

$$X(\hat{n})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{lm}^{X}Y_{lm}(\hat{n}) \quad \Rightarrow \quad a_{lm}^{X} \quad = \quad \int d^2\hat{n}\,X(\hat{n})\,Y_{lm}^{*}(\hat{n})\,, \tag{8.15}$$

where $X=\{\Delta T,E\}$. The isotropy of the universe tells us that different values of m correspond to independent realisations of the universe. Using the ergodic theorem (see Lecture), we can then approximate quantum or stochastic averages, which we can compute from the theory side with angular averages, which can be observed experimentally

$$\langle\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2\dots\rangle \quad \sim \quad \frac{1}{(2l+1)}\sum_m a_{lm}^{\mathcal{O}_1}a_{lm}^{\mathcal{O}_2}\dots\,. \tag{8.16}$$

$$\text{theory} \quad \leftrightarrow \quad \text{observations} \tag{8.17}$$

For example, the correlation between ΔT and E can be obtained observationally from the observed spherical harmonic coefficients

$$\langle a_{lm}^{T}a_{lm}^{E}\rangle=\frac{1}{(2l+1)}\sum_m a_{lm}^{T}a_{lm}^{E}\equiv C_l^{TE}\,. \tag{8.18}$$

It is customary to plot the quantity $\mathcal{D}_l^{ET}\equiv l(l+1)C_l^{ET}$ to make the figure more visible. This correlation was measured most recently by the Planck satellite is shown in Fig. 17 as function of the multipole l. The green circle draws your attention to the negative cross-correralation for $l\lesssim 100$.

Let us see how we can interpret this feature on the theory side. At cartoonish level, temperature fluctuations are a measurement of dimensionless density fluctuations of the photon-electron-baryon plasma, while the polarization is a measurement of the divergence of the plasma velocity $v(\mathbf{x},t)$ at the spacetime point of origin $(\mathbf{x},t)$ of the CMB photon[65].

$$\frac{\Delta T(\mathbf{x},t)}{\bar{T}}\sim\delta\equiv\frac{\rho(\mathbf{x},t)-\bar{\rho}(t)}{\bar{\rho}(t)}\,, \qquad E(\mathbf{x},t)\sim\partial_i v^i(\mathbf{x},t)\,. \tag{8.19}$$

One therefore finds

$$\langle a_{lm}^{T}a_{lm}^{E}\rangle\sim\langle\delta\,\partial_i v^i\rangle\,, \tag{8.20}$$

We now need to specify the stochastic properties of δ and $\partial_i v^i$, so that we can compute this average. Consider the simplest possible toy model: a single, monocromatic (sound) wave

$$\delta(\mathbf{x},t)=A\cos\left(\mathbf{k}\cdot\mathbf{x}\right)\cos\left(\omega t+\phi\right)\,, \tag{8.21}$$

where ω is some fixed frequency, A is the amplitude and ϕ the phase. To mimic the real calculation we should be doing in a quantum mechanical universe, we will assume that A and ϕ are some random variables drawm from some distribution to be specified. Using the linearised continuity equation

$$\dot{\delta}+\partial_i\left[(1+\delta)\,v^i\right]\simeq\dot{\delta}+\partial_i v^i=0 \quad \text{(fluid continuity eq.)}\,, \tag{8.22}$$

we can compute the velocity as well

$$\partial_i v^i(\mathbf{k},t)=-\dot{\delta}(\mathbf{x},t)=\omega A\cos\left(\cos\left(\omega t+\phi\right)\sin\left(\omega t+\phi\right)\right)\sin\left(\omega t+\phi\right)\,. \tag{8.23}$$

Now we need to assume something about the probability distribution that governs A and ϕ. For this, let us consider the comoving particle horizon at the time the CMB was emitted, the "last scattering" of photon, at redshift $z_{LS}\simeq 1100$. We know from (8.8) that in a decelerating universe this is approximately

[65] This is of course an extreme oversimplification. The different physical effects are discussed a bit more in detail in Box **??**.

the same as the comoving Hubble radius $(aH)_{LS} \simeq 4 \times 10^{-3}$ Mpc^{-1}, corresponding to CMB multipoles of approximately $l_{LS} \sim \tau_0 k_{LS} \simeq 70$. Therefore observations on $l \lesssim l_{LS}$ effectively measure perturbations that were super-horizon at the time of emission. In addition, perturbations with $l_{LS} < l < 150$ have spent less than one Hubble time H^{-1} inside the Hubble radius. Since their typical frequency is also of the order of H, they have evolved little from their initial value on superHubble scales. It seems then reasonable to assume that the distribution of ϕ is not peaked around any specific value, since no causal process could have chosen one over another. We will then tentatively assume a flat distribution for $\phi \in \{0, 2\pi\}$, i.e. incoherent, uncorrelated phases. Then the cross-correlation vanishes,

$$\begin{aligned} \langle \delta \partial_i v^i \rangle &\propto \langle AA \rangle \langle \cos(\omega t + \phi) \sin(\omega t + \phi) \rangle & (8.24) \\ &= \langle AA \rangle \int_0^{2\pi} d\phi \cos(\omega t + \phi) \sin(\omega t + \phi) = 0 \,, & (8.25) \end{aligned}$$

where non-random variables such as ω and $\cos(\mathbf{k} \cdot \mathbf{x})$ can be factored outside the average. Since this correlator was our proxy for C_l^{TE}, which is instead observed to be negative and far from zero in Fig. 17, we conclude that the initial superHubble phases were not random but rather *coherent*. In other words, any two perturbations (with the some fixed wavenumbers $|\mathbf{k}| = |\mathbf{k}'|$ corresponding to the same l) must have been been synchronised at some early time before the Hot Big Bang.

One last piece of evidence as to how the synchronisation might have taken place is the negative sign of the correlation. Gravitational collapse is often quoted to make "the rich richer and the poor poorer". This alludes to the fact that, when pressure is negligible, the leading (growing) mode of linearized gravitational collapse consists of a flow away from underdense regions into overdense ones. In formulae

$$\delta > 0 \quad \Rightarrow \quad \dot{\delta} > 0 \quad \Rightarrow \quad \partial_i v^i \sim -\dot{\delta} < 0 \,, \tag{8.26}$$

and viceversa, where in the last step I used the (non-relativistic, linear) continuity equation[66]. This is pictorially summarized in Fig. **??**. Notice that, even if one started with some different initial conditions, say with completely uncorrelated δ and $\partial_i v^i$, always in the absence of pressure, this mode will eventually dominate. Therefore, we would not be surprised to find anti-correlations on scales that have spend some sizable amount of time evolving inside the Hubble radius in the absence of pressure. On the other hand, the negative ET correlation on large scales, $l < 150$, tells us that the coherent superHubble perturbations where already in the "growing" mode, even though there was not enough time for any late-time dynamics to select this mode. Some sort of gravitational collapse mush have started already in the very early universe.

8.2.2 Scale invariance problem*

The second an last problem with the perturbed universe is the surprising fact that the amplitude of perturbations observed in our universe is approximately the same (within 4%) on all cosmological scales (about 3 orders of magnitudes $10^{-4} - 10^{-1}$ Mpc-1). This remarkable feature of what we can now call *primordial perturbations* goes under the name of (approximate) *scale invariance*[67] . The mathematical statement is that for every $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}^+$, a field ϕ obeys scale invariance iff[68]

$$\langle \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\ldots\phi(\mathbf{x}_n)\rangle = \langle \phi(\lambda\mathbf{x}_1)\phi(\lambda\mathbf{x}_2)\ldots\phi(\lambda\mathbf{x}_3)\rangle \,, \tag{8.28}$$

where all the fields are evaluated at the same time[69]. Scale invariance is most evident in the large scales ($l \lesssim 40$) of the CMB temperature angular power spectrum, i.e. the average (or quantum correlator)

$$C_l^{TT} \equiv \frac{1}{2l+1} \sum_l a_{lm}^T (a_{lm}^T)^* = \langle a_{lm}^T (a_{lm}^T)^* \rangle \,. \tag{8.29}$$

[66] cfu: *Check that the addition of the linear relativistic correction (e.g. in Newtonian gauge) does not alter the sign of* $\partial_i v^i$.

[67] cfu: *Primordial perturbations are most easily discussed in terms of the curvature perturbation $\mathcal{R}$, which are time independent on superHubble scale. In this sense, the initial conditions can be though of as correlators in a $(0+3)$-dimensional field theory. In this Euclidean interpretation correlators are fully conformal invariant*

[68] CFU: *Derive the equivalent statement for the correlators of the Fourier transform of the field $\phi(\mathbf{k})$. In particular, for the two-point function in Fourier space, a.k.a. the power spectrum, you should find*

$$\langle \phi(\mathbf{k})\phi(\mathbf{k}')\rangle = (2\pi)^3 \delta_D\left((\mathbf{k}+\mathbf{k}'\right) \frac{C}{k^3} \,, \tag{8.27}$$

for some constant C.

[69] Beware that this is Cosmology lingo. In other fields, such as Conformal Field Theory, sometimes the term scale invariance is used to refer to the invariance under scaling of time as well as space in the correlators.

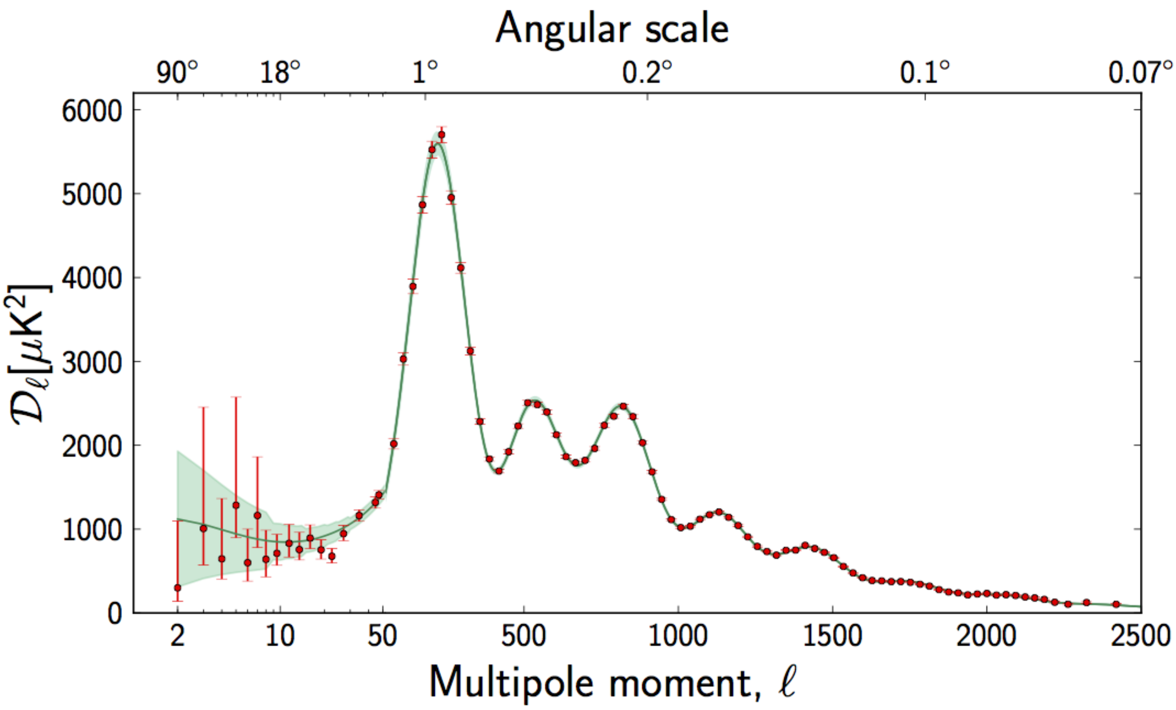


Figure 18:

From Fig. 18, we see that on large scales or small multipoles $l \ll 70$, where we can neglect the acoustic oscillations of the photon-electron-baryon plasma (to be discussed in Lecture P.9.4), the angular power spectrum C_l is well approximated by $\mathcal{D}_l = l\,(l+1)\,C_l = \text{const}$.

By using and abusing the flat sky approximation, [70] one finds

$$\langle \delta T(\hat n)\delta T(\hat n')\rangle \simeq \int dl^2 dl'^2\, e^{i\left(\mathbf{l}\cdot\mathbf{n}+\mathbf{l}'\cdot\mathbf{n}'\right)}\,\langle \delta T(\mathbf{l})\delta T(\mathbf{l}')\rangle \tag{8.31}$$

$$\simeq \int dl^2 dl'^2\, e^{i\left(\mathbf{l}\cdot\mathbf{n}+\mathbf{l}'\cdot\mathbf{n}'\right)}\,\langle a(\mathbf{l}) a(\mathbf{l}')\rangle \tag{8.32}$$

$$\simeq \int dl^2\, e^{i\mathbf{l}\cdot\left(\mathbf{n}-\mathbf{n}'\right)}\,C_l\,. \tag{8.33}$$

Since $C_l \sim l^{-2}$, one recognises in the last line the solution of Poisson's equation[71] with a uniform constant source. By appropriately regulating the divergence, the solution is a constant, i.e. independent of $\mathbf{n}-\mathbf{n}'$, so the primordial correlation function of $\mathcal{R}$ is independent of scale (distance $|\mathbf{n}-\mathbf{n}'|$) as advertised. An analogous derivation goes through using the large scales of the matter power spectrum (see right panel of Fig. P.9.4), but I leave this to the ambitious reader.

One would like to see scale invariance emerging from some (scaling) symmetry of the primordial physics that generated perturbations. A very simple and elegant solution is found by assuming that, during some primordial era, the background spacetime was well approximated by *de Sitter space* (dS) in flat slicing (see Sec. 8.3)

$$ds^2 = \frac{-d\tau^2 + dx^i dx^j \delta_{ij}}{\tau^2 H^2}\,, \tag{8.34}$$

for some constant Hubble parameter H.

[70] cfu: *The flat sky approximation corresponds to the substitution*

$$\frac{\delta T}{\bar T}(\hat n) = \sum_{lm} a_{lm} Y_{lm}(\hat n) \to \Theta(\mathbf{n}) = \int dl^2\, e^{i\mathbf{l}\cdot\mathbf{n}}\,\Theta(\mathbf{l})\,, \tag{8.30}$$

where the coordinates of the sphere $\hat n = \{\theta,\phi\}$ are approximated by euclidean 2d coordinates $\mathbf{n} = \{n_1, n_2\}$. This is valid as long as we consider only a small portion of the sphere (sky).

[71] cfu: *The mathematically inclined reader can proceed to perform the integral directly by using polar coordinates and the residue theorem. It is useful to include a small tilt $C_l \propto l^{-2+\epsilon}$ to regulate the result.*

Box 8.3 Invariance under translations and rotations Consider the most general homogeneous and isotropic spaces, namely an FLRW space. If all other relevant background quantities are also homogeneous and isotropic, then all primordial correlators must be left invariant by the generators of spatial translations and rotations. In real space, these are

$$P_i : -\partial_i \quad \text{and} \quad R_{ij} : -(x_i\partial_j - x_j\partial_i) \,, \tag{8.35}$$

and act on the argument of each perturbation ϕ (assumed to be a scalar for simplicity) as in

$$\sum_{a=1}^{n} \frac{\partial}{\partial \mathbf{x}_a} \langle \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\ldots\phi(\mathbf{x}_n)\rangle \overset{!}{=} 0\,, \tag{8.36}$$

$$\sum_{a=1}^{n} \left(x_a^i \frac{\partial}{\partial x_a^j} - x_a^j \frac{\partial}{\partial x_a^i} \right) \langle \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\ldots\phi(\mathbf{x}_n)\rangle \overset{!}{=} 0\,. \tag{8.37}$$

The general solution of the first constraint is that the correlator only depends on $n-1$ variables, for example $\mathbf{x}_a - \mathbf{x}_1$ for $a = 2, \ldots n$. The generators acting on Fourier space correlators are

$$P_i : -k_i \quad \text{and} \quad R_{ij} : -(k_i\partial_j - k_j\partial_i) \,, \tag{8.38}$$

and therefore

$$\sum_{a=1}^{n} \mathbf{k}_a \langle \phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\ldots\phi(\mathbf{k}_n)\rangle \overset{!}{=} 0\,, \tag{8.39}$$

$$\sum_{a=1}^{n} \left(k_a^i \frac{\partial}{\partial k_a^j} - k_a^j \frac{\partial}{\partial k_a^i} \right) \langle \phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\ldots\phi(\mathbf{k}_n)\rangle \overset{!}{=} 0\,. \tag{8.40}$$

The first condition is satisfy if the correlator is proportional to Dirac delta function, while the second requires it to depend only on the rotational invariant contractions $\mathbf{k}_a \cdot \mathbf{k}_b$ EP: *What about ϵ_{ijk}?*.

One of the ten isometries of this maximally symmetric spacetime is the *dilation* symmetry[72]

$$\tau \to \lambda\tau\,, \quad \mathbf{x} \to \lambda\mathbf{x}\,. \tag{8.41}$$

If all other non-gravitation background quantities depend very weakly on time, then Eq. (8.41) is an approximate symmetry of the full theory and primordial correlators must be invariant under it. Following [?], it is then immediate to see scale invariance arise. In Fourier space, under the transformation Eq. (8.41), a field scales as $\phi(\mathbf{k}, \tau) \to \phi(\mathbf{k}/\lambda, \lambda\tau)$ so the power spectrum must take the form in Eq. (8.27) up to an arbitrary function $F(k\tau)$, which must be zero if the field under consideration is constant, as it is the case for $\mathcal{R}$ on superHubble scales.

It is useful to prove this simple result using a more cumbersome but also more powerful formalism. It is easiest work again in Fourier space and introduce the following notation

$$\langle \phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\ldots\phi(\mathbf{k}_n)\rangle = (2\pi)^3\, \delta_D \left(\sum_{b=1}^{n} \mathbf{k}_b \right) \langle \phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\ldots\phi(\mathbf{k}_n)\rangle' \,. \tag{8.42}$$

Then the generator of dilations in real space[73] is

$$D: \ -\tau\partial_\tau - x^i\partial_i \quad (\text{real space})\,, \tag{8.44}$$

acting on *each* field in the correlator. When acting on primed Fourier-space correlators $\langle\ldots\rangle'$, the generator becomes[74]

$$D: \ -3 + \sum_{a=1}^{n} (3 - \tau_a\partial_{\tau_a}) + k_a \frac{\partial}{\partial k_a} \quad (\text{Fourier space})\,. \tag{8.45}$$

[72] CFU: *Prove this assertion using the definition Eq. (3.1)*

[73] CFU: *Check that indeed $\xi^\mu = \{-\tau, -x^i\}$ is a Killing vector for the dS metric in Eq. (8.41), namely it solves*

$$\mathcal{L}_\xi g_{\mu\nu} = -(\nabla_\mu \xi_\mu - \nabla_\mu \xi_\mu) = 0\,. \tag{8.43}$$

where $\mathcal{L}$ is the Lie derivative. Convince yourself that this equation is equivalent to Eq. (3.1).

[74] CFU: *If you desire reproducing this, keep in mind that the -3 in front comes from the Dirac delta I factored out in Eq. (8.42), the $+3$ comes from the Fourier transform in each coordinate and I used the identity $\mathbf{k} \cdot \partial_{\mathbf{k}} = k\partial_k$.*

The desired scale invariance is obtained by requiring that D leaves correlators of $\mathcal{R}$ invariant. Since *these*$\mathcal{R}$ is conserved on superHubble scales, we can drop the time derivatives and find

$$\left[3(n-1)+\sum_{a=1}^{n}k_a\frac{\partial}{\partial k_a}\right]\langle\mathcal{R}(\mathbf{k}_1)\mathcal{R}(\mathbf{k}_2)\ldots\mathcal{R}(\mathbf{k}_n)\rangle'\stackrel{!}{=}0\,. \tag{8.46}$$

For the power spectrum[75] $P_{\mathcal{R}}(k)\equiv\langle\mathcal{R}(\mathbf{k})\mathcal{R}(\mathbf{k}')\rangle'$, this gives

$$\left[3+k\frac{\partial}{\partial k}+k'\frac{\partial}{\partial k'}\right]P_{\mathcal{R}}(k)\stackrel{!}{=}0\quad\Rightarrow\quad P_{\mathcal{R}}(k)=\frac{C}{k^3}\,, \tag{8.47}$$

for some constant C. Summarizing, the observed scale invariance of the primordial power spectrum follows directly from the dilation isometry of de Sitter space.

8.3 de Sitter spacetime

De Sitter spacetime (dS) is one of three maximally symmetric spacetimes, together with Anti-de Sitter (AdS) and Minkowski space. Recall from Lecture 3, that maximally symmetric spaces in D spacetime dimensions have $D(D+1)/2$ isometries[76]. Therefore, in our $(3+1)$-dimensional world, dS has 10 Killing vectors. It arises as a solution of Einstein equations in the presence of a cosmological constant

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda=0\,. \tag{8.48}$$

The trace of this expression (for $d>2$) tells us $R=\Lambda 2d/(d-2)$ and therefore dS is an Einstein manifold, namely the Ricci tensor is proportional to the metric[77]

$$R_{\mu\nu}=\frac{2\Lambda}{d-2}g_{\mu\nu}\,. \tag{8.50}$$

dS in D-dimensions can be defined as a codimension one, hyperbolic surface in $(D+1)$-dimensional Minkowski space, defined by[78]

$$-\left(X^0\right)^2+\sum_{a=1}^{d}X^aX^a=L^2\,,\quad\text{with}\quad\Lambda=\frac{(d-2)\,(d-1)}{2L^2}\,, \tag{8.51}$$

where L is the dS radius. The dS hyperboloid is invariant under $(D+1)$-dimensional Lorentz transformations (but not translations), namely the group[79] $SO(D,1)$. While the $(D+1)$ Minkowski coordinates of Eq. (8.51) are useful because they transform linearly under this $SO(D,1)$ isometry group, they are clearly redundant. There are three common ways to define D non-redundant coordinates (see [45] for other useful coordinates), which differ in how dS is sliced into constant time hypersurfaces. All three slicings can be though of as intersecting the dS hyperboloid in Eq. (8.51) with a one-parameter family of D-dimensional hyperplanes:

- If the vector perpendicular to the planes is time-like with respect to the $(D+1)$ metric, namely the planes are more "horizontal" than 45 degrees, their intersection with the hyperboloid has a finite volume. Without lost of generality, one can choose the planes to be horizontal (the circles on the lefthand side of Fig. 19). This is called *closed slicing* of dS because the constant time hypersurfaces of dS are hyper-spheres, with positive spatial curvature and finite volume.
- Analogously, a family of "vertical" planes provides the *open slicing*, with constant-time hypersurfaces of negative spatial curvature and infinite volume.

---

[75] CFU: *Using Eq. (8.46) derive the scaling of any n-point function.*

[76] cfu: *This is easily remembered as the dimension of the d-dimensional Poincaré group $\mathbb{R}^{(d-1,1)}\times SO(d-1,1)$ or as that of the $(d+1)$-dimensional Lorentz group $SO(d,1)$.*

[77] Actually, the full Riemann tensor is also given in terms of the metric

$$R_{\mu\nu\rho\sigma}=\frac{R}{12}\left(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}\right)\,. \tag{8.49}$$

[78] cfu: *Notice that the sign of $+L^2$ is such that the hyperboloid lies outside the light cone ($X^a=0$ yields no solution)*

[79] cfu: *How many boost and how many rotations?*

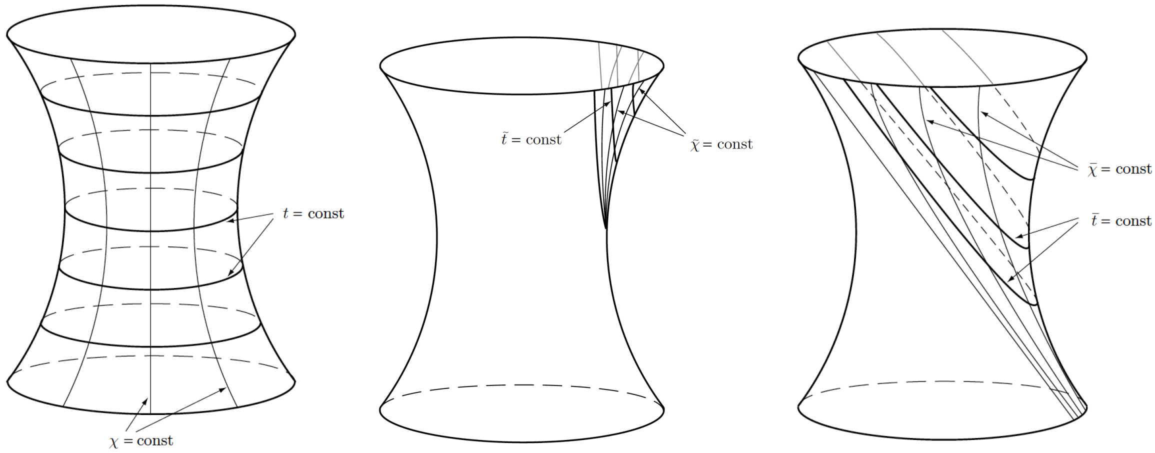


Figure 19: The three time slicing of dS space (from [35]). From left to right they are closed, open and flat slicing.

- The case in between, namely 45 degrees planes, has flat constant-time hypersurfaces of infinite volume. This slicing is commonly used for inflation, which dilutes curvature and makes it negligibly small. The flat-slicing metric in normal and conformal time is[80]

$$ds^2 = -dt^2 + e^{2Ht}dx^2 = \frac{-d\tau^2 + dx^2}{\tau^2 H^2}\,, \tag{8.52}$$

related to the Minkowski coordinates by (here $i = 1, \ldots, d-1$)

$$X^0 = L\sinh(Lt) - \frac{1}{2}\frac{x^i x_i}{L}e^{-Lt}\,, \quad X^i = x^i e^{-Lt}\,, \quad X^d = L\cosh(Lt) - \frac{1}{2}\frac{x^i x_i}{L}e^{-Lt}\,. \tag{8.53}$$

Finally, it is useful to consider combination of dS coordinates that are invariant under dS isometries. The simplest one requires two points and can be thought of as an *invariant distance*. Using the (redundant) $(D+1)$ Minkowski coordinates, this distance is obviously

$$|X - X'|^2 = (X - X')^\mu\, \eta_{\mu\nu}\, (X - X')^\nu\,, \quad (\mu = 0, 1, \ldots, d)\,. \tag{8.54}$$

Since the two points X and X' lie on the dS hyperboloid, $|X|^2 = |X'|^2 = L^2$, so the only part of this distance that actually depends on their position is $X^\mu \eta_{\mu\nu} X'^\nu$. It is therefore convenient to define the invariant distance as

$$\begin{aligned}
D(X;X') &\equiv -X^0 X'^0 + X^i X'^i \quad (i = 1, \ldots, d)\,, &(8.55)\\
D(t, x^i; t', x'^i) &\equiv \cosh(Ht - Ht') - \frac{|x - x'|^2}{2H^2} e^{-H(t+t')}\,, &(8.56)\\
D(\tau, x^i; \tau', x'^i) &\equiv \frac{\tau^2 + \tau'^2 - |x - x'|^2}{2\tau\tau'}\,, &(8.57)
\end{aligned}$$

for the different sets of coordinates.

Box 8.4 Penrose diagrams ...

LESSON 9

Single-field slow-roll inflation

ref

The problems encountered in the previous section suggested we need a prolonged phase of accelerated expansion (curvature, horizon and phase coherence problem), with a background close to dS, which we

[80] CFU: *Derive the relation between the Hubble parameter H and the dS radius L.*

will call *inflation* [21]. In this section, I move beyond these kinematical considerations and discuss the dynamics of inflation.

As we saw in the previous section around Eq. (8.48), a cosmological constant Λ supports a dS solution. However, as the name suggest, the cosmological *constant* does not change with time and the dS phase would be eternal, and could not be connected to the universe as we know it. There is an easy fix: let us introduce a clock ϕ that "turns off" Λ after some time so that the dS phase can indeed stop when desired. I will call this clock-dependent cosmological non-constant $V(\phi)$, to avoid confusing it with the cosmological constant Λ. We can now proceed in two different directions:

1. We can simply specify some function $\phi(t)$ and obtain the desired inflationary background. Naively, this breaks explicitly the diffeomorphism invariance upon which GR is build and seems to introduce a time-dependent function by hand. On a second thought, a gauge symmetry[81] can never be really broken (as the Stückelberg trick teaches) and the choice of time in GR is arbitrary anyways. This approach, made popular by [7], is very effective (pun intended) for model-independent discussion, to highlight the role of symmetries and finally to make connection with observations. On the other hand, it requires a higher level of abstraction than the alternative.

2. We can insist that $\phi(t)$ is the solution of some diff-invariant theory. The simplest choice, as we will see shortly, is a single, canonical scalar field minimally coupled to gravity. An advantage of this point of view is that it provides an important stepping stone to understand the origin of inflation within a UV-complete theory of gravity, such as string theory. This second approach is more intuitive and pedagogical, and so more appropriate for this introductory course.

9.1 Prolonged quasi-de Sitter expansion

The horizon, curvature and phase coherence problems taught us that we should postulate the existence of an early phase of accelerated expansion $\ddot{a}, \dot{a} > 0$, which we call *inflation*. Let us reformulate this as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = H^2 (1 - \epsilon) > 0 \,, \tag{9.1}$$

where I have introduced the *first Hubble slow-roll parameter*

$$\boxed{\epsilon \equiv -\frac{\dot{H}}{H^2}} \,, \tag{9.2}$$

which is a dimensionless measure of the time variation of H. Using (3.53) it is easy to see that for a single-fluid universe $\epsilon = 3(1+w)/2$. From (9.1), we recognise that acceleration requires $\epsilon < 1$ (or $w < -1/3$, as we knew from (3.52)). Also, as long as the matter sector satisfies the Null Energy Condition (see Box 2), $\epsilon > 0$ (or $w > -1$). Observations of both of the CMB and of Large Scale Structures (LSS) probe cosmological scales over roughly three orders of magnitudes[82]and observe approximate scale invariance up to percent corrections (see the scale invariance problem P.9.4). A detailed study of cosmological perturbations (see Lecture P.9.4) shows that scale invariance follows very generically if the spacetime background during inflation is close to de Sitter spacetime, i.e. H is approximately constant. Quantitatively, we will therefore be interested in $0 < \epsilon \ll 1$ (or $w \sim -1$) during inflation.

Let us estimate how long inflation has to last to address the problems discussed in the previous section. A necessary condition to solve the horizon problem is that the particle horizon is larger than the observable universe today. In terms of comoving quantities

$$x_{\text{p.h.}} > r_H = \frac{1}{a_0 H_0} \quad \text{(horizon problem)} \,. \tag{9.3}$$

It is convenient to multiply both sides by the Hubble radius at the end of inflation. This is the time when the early acceleration expansion stops and the decelerated hot Big Bang starts. We will call this time

[81] cfu: *GR can indeed be thought of as a gauge symmetry, with spacetime varying Lorentz transformations.*

[82] cfu: *On the large-scale end, both CMB and LSS probe subHubble scales (although LSS surveys up to date have still a rather small volume and so give weaker constraints than CMB on the largest scales). On the short-scale end, CMB anisotropies are cut-off by the thickness of the last scattering surface and diffusion (a.k.a. Silk-) damping to scales of about* $.2\times$ *Mpc*$^{-1}$. *LSS in principle extend to shorter scales, but our lack of understanding of non-linear and baryonic physics limits our current ability to extract cosmological information from scales smaller than about* $0.2\times$ *Mpc*$^{-1}$. *To currently both CMB and LSS probe a similar window of scales* $\{10^{-4} - 10^{-1}\}$ *Mpc*$^{-1}$. *There is hope to enlarge this "CMB/LSS window" towards smaller scales with the CMB spectrum and 21 cm.*

reheating since this is when the energy is transferred from the inflationary sector to Standard Model particles. If we indicate the comoving Hubble radius by $r_{H_{\rm reh}} = (a_{\rm reh}H_{\rm reh})^{-1}$ and use (8.12) for the particle horizon during a quasi de Sitter expansion, we find

$$\frac{a_{\rm reh}H_{\rm reh}}{a_i H_i} > \frac{a_{\rm reh}H_{\rm reh}}{a_0 H_0}\,, \tag{9.4}$$

where a_i indicates the beginning of inflation. There is great uncertainty about the time of reheating. We are going to parameterize this uncertainty using the temperature of the plasma of Standard Model particle at that time

$$3M_{\rm Pl}^2 H_{\rm reh}^2 = g_* \frac{\pi^2}{30} T_{\rm reh}^4\,, \tag{9.5}$$

where $g_* \sim 100$, but the precise value will not matter given the much large uncertainty in $T_{\rm reh}$. Also, since the temperature of photon has approximately evolve at $T \sim 1/a$ until now[83], we can estimated $a_{\rm reh} \sim T_{CMB,0}/T_{\rm reh}$. Then the right-hand side of (9.4) is

$$\frac{a_{\rm reh}H_{\rm reh}}{a_0 H_0} \simeq 4\times 10^{21}\left(\frac{T_{\rm reh}}{10^{10}{\rm GeV}}\right)\,. \tag{9.6}$$

The actual reheating temperature may dramatically differ from the reference temperature 10^{10}GeV, and a reasonable range of uncertainty is $T_{\rm reh} \in \{1 - 10^{15}\}$ GeV. It is convenient to re-express the duration of inflation on the left-hand side of (9.4) in terms of *efoldings* of expansion, defined by

$$dN \equiv \frac{da}{a} = H dt \quad \Rightarrow \quad N_2 - N_1 = \log\left(\frac{a_2}{a_1}\right)\,. \tag{9.7}$$

Taking the log of (9.4) we finally find

$$\Delta N_{\rm infl} > 50 + \log\left(\frac{T_{\rm reh}}{10^{10}{\rm GeV}}\right)\,, \tag{9.8}$$

and so $\Delta N_{\rm infl} \in \{25 - 60\}$. I'll often use $\Delta N_{\rm infl} \sim 50$ for numerical estimates.

We observe approximate scale invariance for about 7 of the total $\Delta N_{\rm infl}$ efoldings of expansion, but it is natural to assume that $\epsilon \ll 1$ remains to be valid during most of inflation. To quantify this, let us re-write the definition of ϵ and generalise it to the second and higher order Hubble slow-roll parameters[84]

$$\begin{aligned}
\epsilon &\equiv -\frac{\dot H}{H^2} = -\partial_N \ln H\,, && (9.9)\\
\eta &\equiv \frac{\dot\epsilon}{H\epsilon} = \partial_N \ln(\epsilon)\,, && (9.10)\\
\xi_{n\geq 3} &\equiv \partial_N \ln \xi_{n-1}\,, && (9.11)
\end{aligned}$$

with $\xi_2 \equiv \eta$ and where I used $dN = Hdt$ from (9.7). Then, the Taylor expansion of ϵ around some reference time N_* is

$$\begin{aligned}
\epsilon(N) - \epsilon(N_*) &= \left.\frac{\partial\epsilon}{\partial N}\right|_{N_*}(N-N_*) + \left.\frac{\partial^2\epsilon}{\partial N^2}\right|_{N_*}\frac{(N-N_*)^2}{2} + \mathcal{O}\left(\partial_N^3\epsilon\right) && (9.12)\\
&= \epsilon\left[\eta\,(N-N_*) + \eta\xi_3\frac{(N-N_*)^2}{2} + \mathcal{O}\left(\eta^3, \eta^2\xi_3, \eta\xi_3\xi_4, \epsilon\right)\right]\,, && (9.13)
\end{aligned}$$

where all the slow-roll parameters are evaluated at N_*. The requirement that ϵ does change much during inflation is then $\eta\Delta N_{\rm infl}, \xi_n\eta\Delta N_{\rm infl} < 1$ and so

$$\epsilon, \eta, \xi_n \ll 1 \quad \text{(slow-roll inflation)}\,. \tag{9.14}$$

Note that, under the simplistic assumptions that the Taylor above expansion approximates $\epsilon(N)$ during most of inflation and that $\eta \sim \xi_n$, one can think of η^{-1} as the approximate duration of inflation in efoldings. ask how such e

[83]This neglects the changes in g_* around mass thresholds, but these again lead to small changes in the final result.
[84]cfu: *Notice that all slow-roll parameters are dimensionless.*

9.2 Single field inflation

In the previous subsection, we have characterised the expansion history during inflation. We now want to ask how such an expansion history can emerge dynamically, from solving the equations of motion. To try to mimic a cosmological constant, we were led to consider the action of scalar field coupled to gravity. A minimally coupled[85], canonical (see Box P.9.3) scalar field is the simplest option

$$S = -\int \sqrt{-g}\frac{1}{2}\left[M_{\rm Pl}^2 R + \partial_\mu\phi\partial^\mu\phi + 2V(\phi)\right] , \tag{9.15}$$

where the potential $V(\phi)$ is an arbitrary function. The energy-momentum tensor (2.26) is then[86]

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right] . \tag{9.16}$$

Box 9.1 Non-canonical scalar fields A canonical scalar field has a simple quadratic kinetic term with one spacetime derivative per field, as in (9.15). We easily imagine more general but still covariant possibilities. The most generic one with at most one derivative per field is a generic function $P(X,\phi)$ of ϕ and the kinetic term $X \equiv -\partial_\mu\phi\partial^\mu\phi/2$. The homogeneous equations of motion are then

$$\ddot\phi\left(P_X + 2XP_{XX}\right) + 3H\dot\phi P_X + \left(2XP_{X\phi} - P_\phi\right) = 0 , \tag{9.17}$$

while the Friedmann and acceleration equation read

$$3M_P^2H^2 = 2XP_X - P , \quad -M_P^2\dot H = XP_X . \tag{9.18}$$

These theories can give rise to slow-roll inflation and sometime go under the name of k-inflation [19] or simply "P-of-X " theories. An interesting subclass of these theories are those with an exact "shift symmetry" $\phi \to \phi + c$ resulting in $P = P(X)$, without any ϕ dependence. In flat space these always admit a solution $X = $ const (see Prob. P.9.3) and describe the low-energy effective theory of superfluids [44]. When minimally coupled to gravity, there are no slow-roll solutions [16] but if there is a point X_s where $\partial_X P(X)|_{X_s} = 0$, then there is an exact de Sitter solution (see Prob. P.9.3).

This takes the same form as the energy-momentum tensor of a perfect fluid (see Eq. (2.34)), under the following identifications[87]

$$\rho = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi) , \tag{9.19}$$

$$p = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) , \tag{9.20}$$

$$u_\mu = \frac{\partial_\mu\phi}{\sqrt{-\partial_\mu\phi\partial^\mu\phi}} . \tag{9.21}$$

Let us focus on the homogeneous background dynamics. It is useful to specify the fluid parameterisation to the case $\phi = \phi(t)$,

$$\rho = \frac{1}{2}\dot\phi^2 + V(\phi) , \quad p = \frac{1}{2}\dot\phi^2 - V(\phi) , \quad u_\mu = \{1, \mathbf{0}\} . \tag{9.22}$$

The equation of motion for ϕ following from Eq. (9.15) are simply $\Box\phi = 0$ with the d'Alambert operator defined in Eq. (2.6). It needs to be supplemented with the Friedman equation, Eq. (**??**), to give a closed system of equations. Since we will be interested in accelerated expansion, which dilutes spatial curvature, I will set $K = 0$ in the following. For homogeneous configurations one finds[88]

$$\boxed{\ddot\phi + 3H\dot\phi + V'(\phi) = 0 , \quad \text{(background)}} \tag{9.23}$$

[85] Minimal coupling mean that we should write down a Lorentz invariant Lagrangian and then simply couple it to gravity with the substitutions $d^4x \to d^4x\sqrt{-g}$ and $\partial_\mu \to \nabla_\mu$. This does not capture non-minimal couplings such as for example $Rf(\phi)$ or $R^{\mu\nu\rho\sigma}\partial_\mu\phi\partial_\nu\phi\partial_\rho\phi\partial_\sigma\phi$

[86] CFU: *Compute this from the definition of $T_{\mu\nu}$*

[87] cfu: *Notice that the perfect fluid ansatz, Eq. (2.34), is more general that a single scalar field. For example, how many functions of space (initial conditions) does one need to fully specify a solution $\phi(\mathbf{x},t)$? and how many to specify $\delta(\mathbf{x},t)$ and $\mathbf{u}(\mathbf{x},t)$? Consider carefully the order of time derivatives in the equations of motion of the two systems. As I discuss in Sec. P.9.4, a scalar field maps bijectively to a perfect superfluid rather than a fluid.*

[88] cfu: *Recall that, as consequence of diffeomorphism invariance, Einstein's equations generically imply the equations of motion of matter (see e.g. Sec. 19.6 of [6]). In practice, the Bianchi identities imply the conservation of $T_{\mu\nu}$. Check*

While the first and last terms are very familiar from Newton's law, the middle term[89] represents a genuinely relativistic effect. This is sometimes called *Hubble friction* and always opposes changes in ϕ, slowing down the field. The system is closed using the Friedmann equation

$$\boxed{3H^2 M_{\rm Pl}^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad \text{(background)}\,.} \tag{9.24}$$

For almost any potential these EOMs cannot be solved exactly[90]. On the other hand, as will see shortly, general approximate solutions are available in the regime most relevant for observations (quasi dS). Before proceeding, notice that, by taking the time derivative of Eq. (9.24) and using Eq. (9.23), one finds the very useful *exact* relation

$$-\dot{H} M_{\rm Pl}^2 = \frac{1}{2}\dot{\phi}^2\,. \tag{9.25}$$

Box 9.2 The Hamilton-Jacobi formalism and exact solutions Following [28] and references therein, one can divide both sides of Eq. (9.25) by $\dot{\phi}$ to find

$$2H_{,\phi} M_{\rm Pl}^2 = \dot{\phi}\,, \tag{9.26}$$

where the time dependence of H has been traded for its ϕ dependence, $H(t) = H(t(\phi))$. Then the Friedmann equationEq. (9.23) can be re-written as

$$3H^2 M_{\rm Pl}^2 = V + 2\left(H_{,\phi}\right)^2 M_{\rm Pl}^2\,. \tag{9.27}$$

One can then choose some function $H(\phi)$ and find the potential V form this algebraic equation. The first order differential equation Eq. (9.26) can be solved to find $\phi(t)$ and hence $H(t)$.

9.3 Potential slow-roll parameters

The Hubble slow-roll parameters in (9.9)=(9.11) express in a simple and compact way the necessary requirements of an extended inflationary phase. On the other hand, their dependence on the properties of the scalar field that drives the expansion remains implicit: given some $V(\phi)$, one needs to solve the full dynamics to find $H(t)$. We will now study an approximation scheme to evaluated them more directly.

In the hope to find some easily calculable slow-roll parameters, one might define the *potential slow-roll parameters*

$$\epsilon_V \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2\,, \quad \eta_V \equiv M_{\rm Pl}^2 \frac{V''}{V}\,, \quad \xi_{3V} \equiv M_{\rm Pl}^4 \frac{V'V'''}{V^2}\,, \tag{9.28}$$

and the higher orders will not be relevant for us[91]. The relation between these and the Hubble parameter can be derived by repetitively differentiating the Friedmann equation (9.24) (and using (9.25) and the definition of ϵ)

$$V = (3-\epsilon)\, H^2 M_{\rm Pl}^2 \tag{9.29}$$

with respect to time and using the chain rule $\dot{V} = V'\dot{\phi}$. For example, assuming $\dot{\phi} > 0$ one finds the *exact* expressions (see App. **??** for more relations)

$$\epsilon_V = \frac{\epsilon(\eta - 2\epsilon + 6)^2}{4(\epsilon-3)^2}\,, \quad \eta_V = \frac{\eta(\eta + 2\xi_3 + 6) - 2(5\eta + 12)\epsilon + 8\epsilon^2}{4(\epsilon - 3)}\,. \tag{9.30}$$

---

*these statements for a homogeneous scalar field (you can use Eq. (**??**)). What happens when $\dot{\phi} = 0$? Convince yourself that gravity would erroneously think that $\phi(\mathbf{x}, t) = C$ is a solution for any C, even if $\phi = C$ is not a minimum of the potential. Ponder then on the quote from [34] (Sec. 20.6):*

> *Electromagnetism has the motto, "I count all the electric charge that's here". All that bears no charge escapes its gaze. "I weigh all that's here" is the motto of spacetime curvature. No physical entity escapes this surveillance."*

Apparently, cosmological constants do escape its surveillance.

[89] cfu: *Convince yourself that, unless the numerical coefficient is exactly 3, namely the number of space dimensions, this EOM cannot follow directly from a Lagrangian.*

[90] cfu: *The Hamilton-Jacobi formalism can be used to find the right scalar potential $V(\phi)$ that gives rise to some (restricted) class of exact solutions as discussed in Box 2.*

[91] cfu: *Higher order potential slow-roll parameters can be defined by asking that lower order ones do not change much in one efolding (or one Hubble time).*

Naively it looks like things got even more complicated. But as long as all the Hubble slow-roll parameter appearing here are small, we can find the approximate and much simpler relations

$$\epsilon \simeq \epsilon_V \,, \quad \text{and} \quad \eta \simeq 4\epsilon_V - 2\eta_V \,. \tag{9.31}$$

Notice from their definitions in Eq. (9.28), that the potential slow-roll parameter only depend on $V(\phi)$. This is in general not sufficient to know the solution of the EOM[92], Eq. (9.23), since one still has to impose two initial conditions (ϕ_i and $\dot{\phi}_i$). So what these parameters tell you is that there exist some choice of initial conditions that support and extended phase of inflation, but they do not tell you whether a given solution of the EOM does it or not. In practice, for many classes of potential the inflationary trajectory is a local attractor in phase space, so after some time, the approximation in Eq. (9.31) becomes very good. Beware though that this statement does not hold in general and in principle one needs to consider each case individually.

9.4 Slow-roll inflation

The assumption that slow-roll parameters are small allows to find approximate solutions to the EOM. We will see that the definitions in Eq. (9.28) emerge quite naturally.

For ease of calculation and further convenience, it is useful to introduce a shorter name for the canonical kinetic term

$$X \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \quad \overset{\text{background}}{\longrightarrow} \quad X = +\frac{1}{2}\dot{\phi}^2 \,. \tag{9.32}$$

Then the relevant background equations become

$$\rho = X + V \,, \quad p = X - V \,, \quad \text{and} \quad \dot{X} + 6HX + V'\dot{\phi} = 0 \,, \tag{9.33}$$

where the last equation is just the continuity equation, which is equivalent to the EOM Eq. (9.23) multiplied by $\dot{\phi}$ (see also footnote 88). Making use of Eq. (9.25), the condition $\epsilon \ll 1$ tells us that the Friedmann equation, Eq. (9.24), is dominated by the potential term V and we can neglect the kinetic term X,

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{X}{H^2} = \frac{3X}{V + X} \ll 1 \quad \Rightarrow \quad X \ll V \,, \tag{9.34}$$

and so

$$3M_{\rm Pl}^2 H^2 \simeq V \,. \tag{9.35}$$

It is then straightforward to derive the *exact* relation

$$\eta = \frac{\dot{\epsilon}}{\epsilon H} = 2\epsilon + \frac{\dot{X}}{XH} \,. \tag{9.36}$$

Since $\epsilon, \eta \ll 1$ we learn that (assuming $\dot{\phi} \neq 0$)

$$\dot{X} \ll XH \quad \Rightarrow \quad 2\ddot{\phi} \ll \dot{\phi} H \,, \tag{9.37}$$

and so we can neglect the acceleration term $\ddot{\phi}$ in Eq. (9.23) (or $\dot{X}$ in Eq. (9.33))

$$3H\dot{\phi} \simeq -V' \,. \tag{9.38}$$

There is a bit more to this equation than meets the eye:

- The second order EOM has become a first order one, which can be straightforwardly integrated (at least formally)
- The righthand side depends only on the shape of the potential, while the lefthand side really knows about the specific solution. This equation is therefore the bridge between Hubble and potential slow-roll parameters[93].

[92] cfu: *Take from example a constant potential $V(\phi) = \bar{V}$, so that $\epsilon_V = \eta_V = 0$. The set up some initial $\dot{\phi}_i \neq 0$. Convince yourself that nevertheless ϵ and η can be very large, depending on $\dot{\phi}_i$ and V.*

[93] CFU: *Use Eq. (9.38) and Eq. (9.25) to show that $\epsilon \simeq \epsilon_V$.*

- Third, in this approximate equation, $\dot{\phi}$ is fixed once we specify ϕ. We will see that this remarkable simplification is somewhat an accident of having a single field and does not generalize to two or more fields.

Combining the two approximate equations of motion (9.38) and (9.35) one can reduce the problem to solving the following non-linear 1st order ordinary differential equation

$$\dot{\phi} \sim -\frac{V' M_{\rm Pl}}{\sqrt{3V}} \quad \Rightarrow \quad t = \int d\phi \frac{\sqrt{3V}}{V' M_{\rm Pl}} + \text{const}\,. \tag{9.39}$$

The resulting $\phi(t)$ is the slow-roll solution, which is a good approximation to the exact solution when $\epsilon, \eta \ll 1$. Mountains of papers have been written about the infinitely many possible choices of $V(\phi)$ (see e.g. [29] for an older and [33] for a recent review). I will not review any of them here, but the reader is advised to choose some toy model and work it out in full details, e.g. along the lines of Prob. P.9.1 and Prob. P.9.2.

9.5 End of inflation and reheating

By definition, inflation ends at t_e when $\epsilon(t_e) \geq 1$ and the expansion starts to decelerate. This time can be easily computed if one has an exact solution, whether analytical of numerical. But it is also possible to estimate t_e in the slow-roll approximation, by the condition $\epsilon(t_e) \sim \epsilon_V(\phi_e) = 1$, where $\phi_e = \phi(t_e)$. In many simple models this happens when we approach a minimum of the potential at ϕ_{min}.. It is also possible thought that the potential stops being slow-roll steep and the inflation fast rolls down for some time before settling in a minimum. For consistency with the late universe and the rate of the current acceleration of the universe, one typically assumes that the energy at the minimum matches the cosmological constant today, i.e. $V(\phi_{min}) \sim (10^{-3}\text{eV})^4$. This is such a tiny energy as compared with the typical scale of inflation, (9.5), that we might as well assume $V(\phi_{min}) = 0$ for all practical purposes and ϵ_V generically[94] blows up as we approach it.

Given the above picture, we can estimate the number of efoldings of inflation via the chain rule

$$N = \int dN = \int H dt = \int \frac{H}{\dot{\phi}} d\phi = \int \frac{d\phi}{M_{\rm Pl}\sqrt{2\epsilon}} \simeq \int \frac{d\phi}{M_{\rm Pl}\sqrt{2\epsilon_V}} = \int d\phi \frac{V}{M_{\rm Pl}^2 V'}\,, \tag{9.40}$$

where the integration should run from ϕ_i at the beginning of inflation to ϕ_e at the end where $\epsilon(t_e) \simeq \epsilon_V(\phi_e) \simeq 1$. To help our intuition, let us make the very rough approximation that $\sqrt{2\epsilon_V}$ does not vary much for most of the duration of inflation. Then (9.40) gives the relation

$$\frac{\Delta\phi}{M_{\rm Pl}} \sim \Delta N \frac{M_{\rm Pl} V'}{V}\,. \tag{9.41}$$

This tells us that, to achieve a given number of efoldings, say, $\Delta N \sim 50$, flat potentials need a small field excursion $\Delta\phi = \phi_e - \phi_i$, while steep potential need a large field excursion. It customary to divide inflationary potentials into *small field* or *large field* models, depending on wether $\Delta\phi < M_{\rm Pl}$ or $\Delta\phi > M_{\rm Pl}$, respectively. Then (9.41) tells us that potentials that vary on a parametrically subPlanckian scale $\Lambda_\phi \ll M_{\rm Pl}$, defined as $\Lambda_\phi V' \sim V$, lead to superPlanckian field excursions $\Delta\phi \gg M_{\rm Pl}$ and vice versa. There is an ongoing very active and controversial debate as to whether these large field models are allowed in a consistent quantum theory of gravity.

As the inflaton oscillated around the minimum of the potential, with ever decreasing amplitude due to the Hubble friction term in (9.23), quantum processes become relevant and the inflaton decays into a hot soup of standard model particles.

Problems for lesson 9

P.9.1 Consider the simple "chaotic inflation" potential

$$V(\phi) = \lambda_p \phi^p\,, \tag{9.42}$$

for $p > 0$.

[94] If V is analytic around the minimum, which we can take to be at $\phi = 0$ without loss of generality, we can approximate it with its Taylor expansion and then $\epsilon_V \sim M_{\rm Pl}^2 n^2/(2\phi^2)$, where $n \in 2 \times \mathbb{N}^+$ indicate the first non-vanishing Taylor coefficient at the minimum, usually $n = 2$.

(a) What is the mass dimension of λ_p

(b) When are the potential slow-roll parameters ϵ_V, η_V small?

(c) At what ϕ_e does acceleration end (recall $\ddot{a} > 0 \to \epsilon < 1$)?

(d) Focus on $p = 2$ and find $N(\phi)$ in the slow-roll approximation. What ϕ_i gives 50 efoldings?

P.9.2 Consider a canonically normalized scalar field ϕ with the potential

$$V = V_0 \left[1 + \cos\left(\frac{\phi}{f}\right) \right] , \tag{9.43}$$

with V_0 setting the overall vertical scale and the *axion decay constant* f setting the horizontal scale.

(a) What symmetries does this theory enjoy?

(b) Compute ϵ_V and η_V for this potential, as function of ϕ. Notice how they depend on the overall scale V_0

(c) Estimate ϕ_{CMB} corresponding to 60 efoldings before the end of inflation

(d) In what regime of the parameters f and V_0 does this potential become indistinguishable, during the last 60 efoldings of inflation, from the quadratic potential $m^2\phi^2/2$?

P.9.3 Derive the equations of motion or the $P(X, \phi)$ theories.

(a) Derive the equations of motion (9.17) by varying the action $\delta S/\delta\phi$.

(b) Compute the energy-momentum tensor for homogeneous configurations of the field $\phi = \phi(t)$.

(c) From $T_{\mu\nu}$, compute the energy density ρ and the pressure p.

(d) Derive the Friedmann and acceleration equations (9.18) by using the general expression (3.45) and (3.52), and the expression for ρ and p in terms of $P(X, \phi)$ and its derivatives you computed previously.

(e) Specify to $P = P(X)$ and prove that a stationary point X_s where $\partial_X P(X)|_{X_s} = 0$ provides a solution the EoM. This is called the ghost condensate [3]. What spacetime solution emerges?

P.9.4 Around (9.8) we compute the minimum number of efoldings to solve the horizon problem. Compute the lower bound on $\Delta N_{\rm inf}$ obtained by requiring to solve the curvature problem, assuming that at the beginning of inflation $\Omega_k \lesssim \mathcal{O}(1)$ (but you are allowed to neglect K in the Friedmann equation).

Check for understanding of lesson 9

cfu.9.1 Consider the dynamics of a scalar field with a slow-roll flat potential $\epsilon_V, \eta_V \ll 1$, starting with arbitrary initial conditions. Under what conditions on the initial conditions $\{\phi_i, \dot{\phi}_i\}$ are the Hubble slow-roll parameters small?

cfu.9.2 Was the overall scale of inflation, i.e. V, constrained in some way by the slow-roll requirement? How do the potential slow-roll parameters change under rescaling of V?

cfu.9.3

LESSON 10

Cosmological Perturbation Theory

In this lesson, we sail off the land of calm and homogenous seas into the perilous and stormy spacetime oceans. More specifically we assume[95]

$$
\begin{aligned}
g_{\mu\nu}(x,t) &= \bar{g}_{\mu\nu}(t) + h_{\mu\nu}(x,t)\,, &(10.2)\\
T_{\mu\nu}(x,t) &= \bar{T}_{\mu\nu}(t) + \delta T_{\mu\nu}(x,t)\,, &(10.3)
\end{aligned}
$$

with $|h_{\mu\nu}| \ll |\bar{g}_{\mu\nu}|$, $|\delta T_{\mu\nu}| \ll |\bar{T}_{\mu\nu}|$ and barred quantities representing the homogenous and isotropic exact *background* solutions we discussed in the previous lessons. In particular, $\bar{g}_{\mu\nu}$ is the flat FLRW metric in (3.37), $\bar{T}_{\mu\nu}$ was given in Eq. (3.40) and $\bar{u}^\mu = \{1,0,0,0\}$. We work perturbatively in the small perturbations $|h_{\mu\nu}|$ and $|\delta T_{\mu\nu}|$.

10.1 Linearised equations of motion

In these notes we mostly focus on the leading non-trivial order, namely linear order in $h_{\mu\nu}$ and $\delta T_{\mu\nu}$. We want to expand all equations of motions to *linear order* in perturbations. We start from the two (dependent) set of equations[96]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}\,, \quad T^{\mu\nu}{}_{;\nu} = 0\,. \tag{10.4}$$

The trace reversed EE's are also often useful

$$R_{\mu\nu} = -8\pi G \left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right]\,, \quad T \equiv T^\mu_\mu\,. \tag{10.5}$$

Linearising these equations is lengthy but straightforward. I leave it as an exercise. Nowadays the calculation can be done in less than a second using publicly available codes such as the mathematica package xPand [38] (which uses xAct). I discuss this in Prob. P.10.1. Computing $\delta R_{\mu\nu}$ and $\delta T_{\mu\nu}$ and substituting it into

$$\delta R_{\mu\nu} = -8\pi G \left[\delta T_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\delta T^\lambda_\lambda - \frac{1}{2} h_{\mu\nu}\bar{T}^\lambda_\lambda\right]\,, \tag{10.6}$$

gives the scary looking expressions

$$
\begin{aligned}
-\frac{1}{M_{\rm Pl}^2}\left(\delta T_{ij} - \frac{a^2}{2}\delta_{ij}\delta T^\lambda_\lambda\right) = &-\frac{1}{2}\partial_{ij}h_{00} - \delta_{ij}\left[\left(2\dot{a}^2 + a\ddot{a}\right)h_{00} + \frac{1}{2}a\dot{a}\dot{h}_{00}\right] + \left(H^2 + 3\frac{\ddot{a}}{a}\right)h_{ij}\\
&+ \frac{1}{2a^2}\left(\partial_l^2 h_{ij} - \partial_{l(i}h_{j)l} + \partial_{ij}h_{ll}\right) + \frac{H}{2}\left(\dot{h}_{ij} - \delta_{ij}\dot{h}_{ll}\right) \qquad (10.7)\\
&- \frac{1}{2}\ddot{h}_{ij} + H\delta_{ij}\left(Hh_{ii} + \partial_l h_{0l}\right) + \frac{1}{2}\left(\partial_{(i}\dot{h}_{j)0} + H\partial_{(i}h_{j)0}\right)\,,
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{M_{\rm Pl}^2}\delta T_{j0} = &\, H\partial_j h_{00} + \frac{1}{2a^2}\left(\partial_l^2 h_{j0} - \partial_{jl}h_{l0}\right) + \left(H^2 + 2\frac{\ddot{a}}{a}\right)h_{j0}\\
&+ \frac{1}{2}\partial_t\left[\frac{1}{a^2}\left(\partial_j h_{kk} - \partial_k h_{jk}\right)\right]\,, \qquad (10.8)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{M_{\rm Pl}^2}\left(\delta T_{00} + \frac{1}{2}\delta T^\lambda_\lambda\right) = &\, \frac{1}{2a^2}\partial_l^2 h_{00} + \frac{3}{2}H\dot{h}_{00} - \frac{1}{a^2}\partial_i\dot{h}_{i0} + 3\left(H^2 + \frac{\ddot{a}}{a}\right)h_{00}\\
&+ \frac{1}{2a^2}\left[\ddot{h}_{ii} - 2H\dot{h}_{ii} + 2\left(H^2 - \frac{\ddot{a}}{a}\right)h_{ii}\right]\,, \qquad (10.9)
\end{aligned}
$$

[95] When working with perturbations, one has to decide about the meaning of covariant and contravariant indices (up and down). I use Weinberg's conventions that $\delta T^{\cdots}_{\cdots}$ represents the perturbation of $T^{\cdots}_{\cdots}$, as opposed to the perturbations of say $T_{\cdots\cdots}$ raised by the background metric. For example

$$h^{\mu\nu} = \delta\left(g^{\mu\nu}\right) = g^{\mu\nu} - \bar{g}^{\mu\nu} = -\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}h_{\rho\sigma}\,, \tag{10.1}$$

where I used that the expansion of the inverse of a matrix $M + \delta M$ is $\delta(M^{-1}) = -M^{-1}\delta M M^{-1} + \ldots$. Here $h^{\mu\nu} \neq \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}h_{\rho\sigma}$. Another example is $\delta u^0 = \delta u_0$ in (10.29).

[96] Notice that the different sign in the Einstein Equation depends on convention. I follow Weinberg's notation in this section (different from Dodelson's).

where $\partial_l^2 \equiv \partial_l \delta^{lk} \partial_k$ and (see footnote 95)

$$\delta T^\mu_\nu = \bar{g}^{\mu\lambda} \left[\delta T_{\lambda\nu} - h_{\lambda\rho} \bar{T}^\rho_\nu \right] . \tag{10.10}$$

Notice that, as implied by the Bianchi identities [97] , the four metric perturbations $h_{\mu 0}$ appear with at most one time derivative in these equations and are therefore non-dynamical. It is useful to discuss this quantitatively in terms how many initial conditions we need to and can specify to find a solution. , and are therefore subject to constraints. The linearised conservation of the energy momentum tensor takes the following form

$$\delta \left(\nabla_\mu T^\mu_\nu \right) = \partial_\mu \delta T^\mu_\nu + \bar{\Gamma}^\mu_{\mu\lambda} \delta T^\lambda_\nu - \bar{\Gamma}^\lambda_{\mu\nu} \delta T^\mu_\lambda + \delta \Gamma^\mu_{\mu\lambda} \bar{T}^\lambda_\nu - \delta \Gamma^\lambda_{\mu\nu} \delta T^\mu_\lambda = 0 \,. \tag{10.12}$$

Using (10.10), we can write this in terms of $h_{\mu\nu}$ as

$$\partial_0 \delta T^0_j + \partial_i \delta T^i_j + 2H \delta T^0_j - a^2 H \delta T^j_0 - (\bar{\rho} + \bar{p}) \left(\frac{1}{2} \partial_j h_{00} - H h_{j0} \right) = 0 \tag{10.13}$$

$$\partial_0 \delta T^0_0 + \partial_i \delta T^i_0 + 3H \delta T^0_0 - H \delta T^i_i + \frac{\bar{\rho} + \bar{p}}{a^2} \left(H h_{ii} - \frac{1}{2} \dot{h}_{ii} \right) = 0 \,. \tag{10.14}$$

Three observations make the task of solving the above equations more manageable:

- *Fourier decomposition*: because we expand around a *homogeneous* background, different Fourier modes decouple from each other at linear order.
- *Scalar-Vector-Tensor (SVT) decomposition*: because we work with general covariant theories and we expand around an *isotropic* background, objects that transform differently under spatial rotations do not mix with each other at linear order.
- *Gauge transformations*: since we are dealing with GR, a covariant formulation of gravity, there is some redundancy in our description due to the arbitrary choice of coordinates. One can always perform a coordinate transformation (which we will soon interpret as a gauge transformation on the fields) to conveniently simplify the equations.

Notice that the first two simplifications crucially rely on working at linear order, while the last survives at all orders in perturbation theory. Let us discuss these three points in detail.

10.2 Fourier decomposition

We would like to parameterise the perturbations $h_{\mu\nu}$ and $\delta T_{\mu\nu}$ in such a way as to simplify the calculation as much as possible. Experience teaches us that it is wise to choose variables that transform nicely under the symmetry of the system[98]. While the theories we are working with are fully covariant, i.e. invariant in form under changes of coordinates, the background we have chosen is only invariant under rotations and translations. By rotating and translating a perturbation that solves the equations of motion, we obtain another, in general different perturbation that also solves them. This is a linear operation and so, in more mathematical terms, the space of solutions provides a linear *representation* of the isometry group $\mathrm{SO}(3) \times \mathbb{R}^3 = \mathrm{ISO}(3)$, called the Euclidean group. The building blocks of these representations are irreducible representations (irreps). In this context, an irrep is a family of solutions that can all be transformed into each other by some element of ISO(3). Intuitively[99] these can be thought of as cosmological

[97]To see this recall that the Bianchi identities (2.22) (which are not equations of motion, but indeed identities) say $\nabla^\mu G_{\mu\nu}$ with $G_{\mu\nu}$ the Einstein tensor on the left-hand side of EE's. Expanding the covariant derivative we find

$$\partial_t G^{t\beta} = -\partial_k G^{k\beta} - \Gamma^\alpha_{\alpha\gamma} G^{\beta\gamma} - \Gamma^\beta_{\alpha\gamma} G^{\alpha\gamma} \,. \tag{10.11}$$

Since the right-hand side has at most second derivatives of the metric (in $G_{\mu\nu}$), we conclude that $G^{t\beta}$ has at most first derivatives. It takes a bit more work and the ADM formalism to specify which components of the metric appear with at most one derivative. At linear order, by inspection we see that it is $h_{\mu 0}$.

[98]cfu: *Think about some physical system you have studied and how its symmetries were used to simplify the problem. For example, translational invariance in solid state physics, isotropy in the hydrogen atom or Poincaré invariance in scattering amplitudes.*

[99]This discussion follows closely the analogous introduction of particles in relativistic QFT. In relativistic theories, particles are the irreducible representation of the Poincaré group. These are first classified by their mass $m^2 = -p^\mu p_\mu$. Then, for massive particles $m > 0$, they are further classified by their spin, i.e. the eigenvalues of the total spin operator J^2 and the spin in one of the three spatial directions J_z. Massless particles are instead further classified by their helicity, i.e. the eigenvalue of their angular momentum in their direction of motion $p^i J_i$.

"particles", the building blocks of more general cosmological perturbations. The construction of irreps of the non-compact groupr ISO(3) is easily performed using the method of "induced representations". The idea is to find a representation for a subgroup, in this case the *little group*, and extend that representation to the whole group. A summary of the derivation in Sec. 10.9 is that perturbations are classified by the norm of their three moment $\mathbf{k}^2$ and by their helicities, $0, \pm 1, \pm 2, \dots$. Let us now see how the isometries restrict the possible interaction among these perturbations.

We claim that, because of the homogeneity of the background, different Fourier modes decouple from each other at linear order. To see why this is the case, consider the general form of the linearized equations of motion

$$\sum_A \mathcal{O}_A \,\mathrm{Pert}_A(\mathbf{x}, t) = 0\,, \tag{10.15}$$

where A enumerates all perturbations $\mathrm{Pert}_A = \{h_{\mu\nu}, \delta T_{\mu\nu}\}$ and $\mathcal{O}_A$ are linear differential operators acting on the perturbations (e.g. $H(t)\partial_t$ or $a^{-2}\partial_i\partial_i$). Because of general covariance, these operators must be constructed out of covariant spacetime derivates ∇_μ and other tensorial objects evaluated on the background

$$\mathcal{O}_A = \mathcal{O}_A(\nabla_\mu, \bar{g}_{\mu\nu}, \bar{T}_{\mu\nu}) = \mathcal{O}_A(\partial_\mu, \partial_t^{\#}\bar{g}_{\mu\nu}(t), \partial_t^{\#}\bar{T}_{\mu\nu}(t))\,. \tag{10.16}$$

Since the background is homogeneous, $\mathcal{O}_A$ cannot depend on space $\mathbf{x}$, but it does in general depend on time through $\bar{g}_{\mu\nu}(t)$ and $\bar{\delta}T_{\mu\nu}(t)$. As we take the Fourier transform of (10.15), we find

$$\int d^3x e^{-i\mathbf{x}\mathbf{k}} \sum_A \mathcal{O}_A \,\mathrm{Pert}_A(\mathbf{x}, t) = \sum_A \tilde{\mathcal{O}}_A \,\mathrm{Pert}_A(\mathbf{k}, t) = 0\,, \tag{10.17}$$

with (see Eq. (1.6) for my Fourier conventions)

$$\mathrm{Pert}_A(\vec{k}, t) = \int d^3x\, e^{i\mathbf{x}\mathbf{k}}\,\mathrm{Pert}_A(\mathbf{x}, t) \tag{10.18}$$

$$\text{and}\quad \tilde{\mathcal{O}}_A = \mathcal{O}_A(\partial_t, \partial_i \to ik_i, \partial_t^{\#}\bar{g}_{\mu\nu}(t), \partial_t^{\#}\bar{T}_{\mu\nu}(t)) \tag{10.19}$$

where all spatial derivative have been integrated by part to act on $e^{-i\mathbf{x}\mathbf{k}}$ hence giving $i\mathbf{k}$. While Eq. (10.15) was a *partial* differential equation, Eq. (10.17) is now a infinite set of *ordinary* differential equations, one for each $\mathbf{k}$. Since in each equation only one $\mathbf{k}$ appears in the arguments of Pert_A, different Fourier modes with $\vec{k} \neq \vec{k}'$, decouple from each other. In other words, at linear order one can always look for solutions with a single, monochromatic perturbation with wavevector $\mathbf{k}$ in an otherwise unperturbed background universe. Any linear combination of these solutions is also a solution (linear superposition). Finally, notice that k is the Fourier conjugate of x, and so it is a *comoving* momentum. Physical momentum is instead $k_{\mathrm{phy}} = k/a$, in the same way that $x_{\mathrm{phy}} = xa$.

10.3 Scalar-Vector-Tensor decomposition

Let us know take advantage of the isotropy of the background. Because we work with general covariant theories and we expand around an isotropic background, different helicities, i.e. perturbations that transform differently under *spatial rotations* do not mix with each other at linear order. Let us see why.

Rotations are changes of coordinates of the form

$$\{x^0, x^i\} \to \{x'^0, x'^{i'}\} = \{x^0, R_i{}^{i'} x^i\} \quad \Rightarrow \quad J_\mu^{\mu'} \equiv \frac{\partial x'^{\mu'}}{\partial x^\mu} = \begin{pmatrix} 1 & \\ & R^{i'}{}_i \end{pmatrix} \tag{10.20}$$

and the Jacobian has only non-trivial spacial components. Let us use this to compute how different objects transform. Consider the simplest objects, namely diff-scalars and their perturbations. [100] Some examples are

$$\delta\rho(x,t) \equiv \rho(x,t) - \bar{\rho}(t) \quad \text{and} \quad \delta p(x,t) \equiv p(x,t) - \bar{p}(t)\,, \tag{10.21}$$

[100] Some nomenclature. The terms scalar, vectors and tensor may refer to the transformation of an object either under general change of coordinates, a.k.a. *diffeomorphisms* (diffs), or only under spatial rotations. To be crystal clear, in this section I'll denote these two concepts differently. I define diff-scalars, diff-vectors and diff-tensors objects that transform covariantly under general changes of coordinates, as in (2.4). Analogously, rotations-scalars, rotation-vectors and rotation-tensors will be objects that transform appropriately under rotation, as we will see in the following. In the rest of the lectures instead the difference will hopefully be clear from the context

For general changes of coordinates $x^\mu \to x'^\mu(x)$ any diff-scalar $s = \{\rho, p, \dots\}$ transforms as $s'(x', t') = s(x, t)$. Since rotations are a special case of diffs, we find that perturbations to a diff-scalar are rotation-scalars, i.e. transform as

$$\delta s(x,t) \to \delta s'(x',t) \equiv s'(x',t) - \bar{s}(t) = s(x,t) - \bar{s}(t) = \delta s(x,t)\,. \tag{10.22}$$

Perturbations to diff-vectors (such as u^μ) and symmetric two-tensors (such as $g_{\mu\nu}$) are more complicated. From their transformation properties under general diffeomorphism, (2.4), it is immediate to see that when all indices are in the time direction, these objects transform as rotation-scalars, e.g.

$$\begin{aligned} \delta u'^0(x',t) &\equiv u'^0(x',t) - \bar{u}'^0(t) = J^0_\mu u^\mu(x,t) - \bar{u}^0(t) = u^0(x,t) - \bar{u}^0(t) = \delta u^0(x,t)\,, \\ h'_{00}(x',t) &= g'_{00}(x',t) - \bar{g}'_{00}(t) = J^\mu_0 J^\nu_0 g_{\mu\nu}(x,t) - \bar{g}_{00}(t) = g_{00}(x,t) - \bar{g}_{00}(t) = h_{00}(x,t)\,. \end{aligned}$$

Notice that I only use active transformation for which only perturbations transform, but not the background, e.g. $\bar{u}'^0(t) = u^0(t)$. When the indices are in the spatial directions we can apply the Hodge decomposition, which is a generalization of Helmholtz decomposition, which is familiar from the study of electromagnetism. For example, any spatial vector v^i such as $\delta u^i = u^i$ can be decomposed as[101]

$$v_i = \omega_i + \partial_i \theta\,, \tag{10.23}$$

where ω_i is divergence-free or *transverse*, namely $\partial_i \omega_i = 0$. To find θ, we take the gradient of this equation

$$\partial^i v_i = \nabla^2 \theta\,. \tag{10.24}$$

On a topologically trivial space such as $\mathbb{R}^3$ and assuming that u_i vanishes at spatial infinity, this Poisson equation can be uniquely solved for θ. Then ω_i is simply given by substituting this solution into (10.23).

The Helmholtz decomposition is covariant under rotation if we assume that θ transform as a rotation-scalar (see P.10.2) and ω_i as a rotation-vector, i.e.

$$\omega'_{i'}(x',t) = R_{i'}{}^{i}\omega_i(x,t)\,. \tag{10.25}$$

Exactly the same Helmholz decomposition can be used for any two tensor with one spatial and one time index such as h_{0i}. The last object we will need to decompose is the spatial part of a two-tensor, for example h_{ij}. It is straightforward to see that the trace h^i_i (taken with the background metric) is a rotation-scalar. For the remaining 5 components[102] we can use a generalisation of Helmholz decomposition to any tensor, which breaks up h_{ij} into two rotation-scalars, one transverse vector and a *transverse-traceless* spatial two-*tensor* ($v_{ii} = \partial_i v_{ij} = 0$). The explicit decomposition is given below.

Let us introduce some notation to conveniently deal with the SVT decomposition. The metric perturbation $h_{\mu\nu}$ is a symmetric 4×4 matrix with 10 independent entries. They can be SVT-decomposed as follows[103]

$$\begin{aligned} h_{00} &= -E\,, \\ h_{i0} &= a\left[\partial_i F + G_i\right]\,, \\ h_{ij} &= a^2\left[\delta_{ij}A + \partial_{ij}B + \partial_{(i}C_{j)} + D_{ij}\right]\,, \end{aligned} \tag{10.26}$$

with $\partial_i G_i = \partial_i C_i = D_{ii} = \partial_i D_{ij} = 0$. In P.10.3 you will explicitly perform this decomposition. We have four scalars E, A, B and F, two transverse vectors C_i and G_i (with two "polarizations" each) and one transverse traceless tensor D_{ij} (also two polarizations), adding up to 10, as expected. Analogously, the energy-momentum tensor can be SVT-decomposed as follows (see P.10.3):

$$\begin{aligned} \delta T_{00} &= -\bar{\rho} h_{00} + \delta\rho\,, \\ \delta T_{i0} &= \bar{p} h_{0i} - (\bar{\rho} + \bar{p})\left[\partial_i \delta u + \delta u^V_i\right]\,, \\ \delta T_{ij} &= \bar{p} h_{ij} + a^2\left[\delta_{ij}\delta p + \partial_{ij}\pi^S_{ij} + \partial_{(i}\pi^V_{j)} + \pi^T_{ij}\right]\,, \end{aligned} \tag{10.27}$$

[101] Helmholtz theorems states that any smooth and rapidly decreasing at infinity scalar field can be uniquely decomposed into a curl-free vector and a divergence-free vector. In $\mathbb{R}^3$, these vectors can be written as the gradient of a scalar potential potential (e.g. the electro-static potential) and the curl of a vector potential (e.g. the vector potential generating a magnetic field). In cosmology it is customary to work with the scalar potential (e.g. θ in Eq. (10.23)) and the divergence-free vector (e.g. ω_i in Eq. (10.23)).

[102] 3(3-1)/2 components for a general symmetric tensor minus 1 for the trace.

[103] The factors of a in these definitions are of course arbitrary and chosen for future convenience.

with four scalars ($\delta\rho$, δp, δu and π^S), two transverse vectors (π^V and δu^V) and one transverse traceless tensor (π^T), adding up again to 10. Notice that we SVT decomposed the fluid velocity with a *lower* index

$$u_\mu = \{-1 + \delta u_0, \partial_i \delta u + \delta u_i^V\}\,, \tag{10.28}$$

and that, to maintain the normalization of u^μ, one needs at linear order

$$u_\mu u^\mu = -1 \quad \Rightarrow \quad \delta u_0 = \delta u^0 = h_{00}/2\,. \tag{10.29}$$

The π's are called *anisotropic inertia* and are a property of a given fluid that needs to be specified to close the system of equations. For example, all anisotropic inertia vanishes for a perfect fluid (as can be seen from just counting degrees of freedom in Eq. (2.34)).

Now the essential point: rotation-scalars, rotation-vectors (or transverse vectors) and rotation-tensors (or transverse-traceless tensors) decouple from each other at linear order[104]. The reason is conceptually analogous to the decoupling of different Fourier modes. Intuitively, it is impossible to construct a non-vanishing scalar from a transverse vector ω_i or a transverse-traceless tensor v_{ij} using only derivatives and background quantities. In fact, the only object that one can use to contract the spatial indices are the background spatial metric, proportional to δ_{ij}, and spatial derivatives ∂_i. Any contraction of all indices is identically zero because of the transverse and traceless conditions. A similar argument shows that all other potential mixing terms must vanish.

10.4 Gauge transformations

Since we are dealing with GR, one can always perform a coordinate transformation to simplify the equations. Consider the coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)\,, \tag{10.30}$$

for arbitrary $\epsilon^\mu(x)$. We will be interested in transformations that make some perturbations vanish identically, so we will restrict ourselves to cases in which ϵ^μ is a regular and decreasing function at spatial infinity and it is of first order in perturbations $\epsilon \sim \mathcal{O}(h_{\mu\nu}, \delta T_{\mu\nu})$. While we know that tensors such as $g_{\mu\nu}$ and $T_{\mu\nu}$ transform as in Eq. (2.4), we have now the additional complication that every tensor is split between a background and a perturbation, as e.g. in Eq. (10.2). We have therefore an ambiguity on how the background and the perturbation transform separately, while keeping the covariance of the full tensor. A convenient an very common way to solve this ambiguity is to work with so called *gauge transformations*, in which case the background is kept fixed and all the transformations of the full tensor are attributed to the perturbations. More in detail, the rules are the following

1. Transform the full tensor covariantly, as in Eq. (2.4), but keep the background unchanged
2. Drop the prime from the new coordinates
3. Attribute all the transformation to the perturbations

In equations, for example for a scalar field $s(x) = \bar{s} + \delta s$, one find the transformation $\Delta\delta s$ to be

$$\Delta\delta s \equiv s'(x) - s(x) = s(x-\epsilon) - s(x) = -\epsilon^\mu \partial_\mu s(x) + \mathcal{O}(\epsilon^2)\,, \tag{10.31}$$

where I used

$$s'(x') = s(x) \quad \Rightarrow \quad s'(x) = s(x-\epsilon)\,. \tag{10.32}$$

Since we will always work with a homogeneous background, $\bar{s}(x) = \bar{s}(t)$, this simplifies to

$$\Delta\delta s = -\epsilon^0 \dot{\bar{s}} + \mathcal{O}(\epsilon^2, \epsilon h_{\mu\nu}, \epsilon\delta T_{\mu\nu})\,. \tag{10.33}$$

The same rules apply to vectors and (symmetric two-)tensors, for which one finds (see P.10.4)

$$\begin{aligned} \Delta\delta V^\mu &\equiv V'^\mu(x) - V^\mu(x) = -\epsilon^\nu \partial_\nu V^\mu + V^\nu \partial_\nu \epsilon^\mu = -\epsilon^\nu \nabla_\nu V^\mu + V^\nu \nabla_\nu \epsilon^\mu\,, & (10.34)\\ \Delta h_{\mu\nu}(x) &\equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -\nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu\,. & (10.35)\end{aligned}$$

[104] Decoupling means that in solving the equations of motion for one of the three types of perturbations, I can set the others to zero. Any combination of the three sets of solutions thus obtained is also a solution.

The attentive reader will have noticed that the gauge transformations look very similar to Lie derivatives (see Box 1). This is in fact the case. The transformation of the perturbations to any tensor are given by minus its Lie derivative in the direction ϵ. At linear order this simplifies to

$$\boxed{\Delta\delta\text{Tensor} = -\mathcal{L}_\epsilon \text{Tensor} = -\mathcal{L}_\epsilon \overline{\text{Tensor}} + \mathcal{O}(\epsilon^2)\,,} \tag{10.36}$$

where in the last equality I used the fact that ϵ is already first order in perturbation. In particular, notice that all covariant derivatives in (10.34) should be computed from the background metric. So far we have discussed gauge transformation for the full diffeomorphism tensors, but we would like to know how each SVT component transforms. Using Eq. (10.36) and the SVT decomposition Eq. (10.26) and Eq. (10.27), we find the following linear gauge transformations of the SVT components for the metric[105]

$$\begin{aligned} \Delta A &= 2H\epsilon_0\,, \quad \Delta B = -\frac{2}{a^2}\epsilon^S\,, \\ \Delta C_i &= -\frac{1}{a^2}\epsilon_i^V\,, \quad \Delta D_{ij} = 0\,, \quad \Delta E = 2\dot{\epsilon}_0\,, \\ \Delta F &= \frac{1}{a}\left(-\epsilon_0 - \dot{\epsilon}^S + 2H\epsilon^S\right)\,, \quad \Delta G_i = \frac{1}{a}\left(-\dot{\epsilon}_i^V + 2H\epsilon_i^V\right)\,, \end{aligned} \tag{10.37}$$

and for the energy-momentum tensor

$$\begin{aligned} \Delta\delta\rho &= \dot{\bar{\rho}}\epsilon_0\,, \quad \Delta\delta p = \dot{\bar{p}}\epsilon_0 \quad \Delta\delta u = -\epsilon_0\,, \\ \Delta\pi^S &= \Delta\pi_i^V = \Delta\pi_{ij}^T = \Delta\delta u_i^V = 0\,, \end{aligned} \tag{10.38}$$

where we have used the SVT-decomposed gauge parameter

$$\epsilon^\mu = \{\epsilon^0, \partial^i\epsilon^S + \epsilon_V^i\}\,, \tag{10.39}$$

with $\partial_i\epsilon_V^i = 0$. Notice that the transformations of $\delta\rho$ and δp can be easily derived from Eq. (10.33), and those of δu^μ from Eq. (10.34).

10.5 Vector perturbations

Because we work only with diffeomorphism invariant theories, all equations of motions can be written as the vanishing of some tensor. For example, we are interested in the EE's, i.e. $M_{\rm Pl}^2 G_{\mu\nu} + T_{\mu\nu} = 0$. We can then apply the same SVT decomposition to this 2-index tensor and extract 4 scalar, two transverse-vector and one transverse-traceless-tensor equations. We will start with the vector and tensor equations since they are the simpest and study the scalars last.

Vectors[106] decay with time and so do not play much of a role in cosmology[107]. To see this, let us take advantage of the SVT decomposition and set all scalar and tensor perturbations to zero. We are left with

$$\begin{aligned} &h_{00} = 0 && \delta T_{00} = 0\,, \\ &h_{0i} = aG_i\,, && \delta T_{0i} = \bar{p}aG_i - (\bar{\rho}+\bar{p})\,\delta u_i^V \\ &h_{ij} = a^2\partial_{(i}C_{j)}\,, && \delta T_{ij} = a^2\left(\bar{p}\partial_{(i}C_{j)} + \partial_{(i}\pi_{j)}^V\right)\,, \end{aligned} \tag{10.40}$$

Plugging this into the linearized momentum conservation equation $T^{i\mu}_{;\mu} = 0$, (10.13), one finds

$$\partial^2\pi_j^V + \partial_0\left[(\bar{\rho}+\bar{p})\,\delta u_j^V\right] + 3H\,(\bar{\rho}+\bar{p})\,\delta u_j^V = 0\,. \tag{10.41}$$

All ingredients of the standard cosmological model (baryons, Dark matter, dark energy, photons, neutrinos) behave as a perfect fluid to good approximation and so we neglect the anisotropic inertia[108]. We then find

$$(\bar{\rho}+\bar{p})\,\delta u_j^V \simeq a^{-3}\,. \tag{10.42}$$

[105] Notice that $\epsilon_0 = -\epsilon^0$.

[106] From now on I simply write "vectors" and "tensors" and omit specifying "transverse" and "transverse traceless" every time.

[107] An exception are the speculated primordial magnetic fields (see [] for a review).

[108] Neutrinos do have some anisotropic inertia as they become non-relativistic, and this results in a 10% correction to the spectrum of tensor modes [49]

Using (10.40) into the linearized $0i$ part of the trace-reversed EE's, (10.8), one finds

$$8\pi G\left(\bar{\rho}+\bar{p}\right)\delta u_j^V a=\frac{1}{2}\partial^2\left(G_j-a\dot{C}_j\right)\,, \tag{10.43}$$

and so $G_i - \dot{C}$ decays as a^{-2} by virtue of (10.42). Using Eq. (10.37), one can prove that this combination is indeed the only gauge-invariant vector mode (see Prob. P.10.5).

10.6 Tensor perturbations

From the space-space (ij) components of the EE's one can extract the transverse traceless part following P.10.3. But given that we proved that SVT components decouple, it is much easier to set all scalars and vectors to zero and keep only D_{ij} in the linearised EE's. Substituting

$$h_{0\mu}=0=\delta T_{0\mu}\,,\qquad h_{ij}=a^2 D_{ij}\quad\text{and}\quad \delta T_{ij}=a^2\left(\bar{p}D_{ij}+\pi^T_{ij}\right) \tag{10.44}$$

into (10.7) and recalling that $D_{ii}=\partial_i D_{ij}=0$, one finds (see cfu.10.5)

$$\boxed{\ddot{D}_{ij}+3H\dot{D}_{ij}-\frac{\partial^2}{a^2}D_{ij}=8\pi G\pi^T_{ij}\,.} \tag{10.45}$$

The tensor anisotropic inertia π^T is small for all components of the universe. The largest contributors are neutrinos and their π^T eventually leads to a 20% reduction in the amplitude of D_{ij} (see [49] or Sec 6.6 of [51] for a detailed calculation). After neglecting π^T, (10.45) takes the same form as the equation of motion for a massless scalar field in FLRW (see P.10.6). The solution is best understood in Fourier space

$$\ddot{D}_{ij}+3H\dot{D}_{ij}+\frac{k^2}{a^2}D_{ij}=0\,. \tag{10.46}$$

Polarisation tensors Because of the isotropy of the background, each of the two independent components of D_{ij} has the same time dependence. To make this more explicit, let us separate the index structure from the time dependence:

$$D_{ij}(t,\mathbf{k})=\sum_{s=+,\times}\epsilon^s_{ij}(\mathbf{k})\mathcal{D}_s(t,k)\,. \tag{10.47}$$

Here $\epsilon^{+,\times}_{ij}(\mathbf{k})$ are the "plus" and "cross" *polarisation tensors*, which satisfy the transverse-traceless conditions $k^i\epsilon^s_{ij}(\mathbf{k})=\epsilon^s_{ii}(\mathbf{k})=0$ and the normalisation $\epsilon^s_{ij}\epsilon^{s'}_{ji}=2\delta_{ss'}$. Since all these conditions are invariant under rotations, to find ϵ^s_{ij} explicitly, we can simply choose some convenient $\mathbf{k}$ and then rotate the result. If we take $\hat{\mathbf{k}}=\mathbf{k}/k=\hat{\mathbf{z}}$ a simple solution to all the above conditions is

$$\epsilon^+_{ij}(\hat{\mathbf{z}})=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\quad\text{and}\quad\epsilon^\times_{ij}(\hat{\mathbf{z}})=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} \tag{10.48}$$

This is not the only choice since any rotation around $\hat{\mathbf{z}}$ gives a different choice of polarization. More generally, given any wavevector $\mathbf{k}$, we define $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ to form an orthonormal basis with $\hat{\mathbf{k}}=\mathbf{k}/k$. Then

$$\epsilon^+_{ij}(\mathbf{k})=\hat{u}_i\hat{u}_j-\hat{v}_i\hat{v}_j\quad\text{and}\quad\epsilon^\times_{ij}(\mathbf{k})=\hat{v}_i\hat{u}_j+\hat{v}_j\hat{u}_i\,. \tag{10.49}$$

Time evolution We are not interested in solving

$$\ddot{\mathcal{D}}_s(t,k)+3H\dot{\mathcal{D}}_s(t,k)+\frac{k^2}{a^2}\mathcal{D}_s(t,k)=0\,. \tag{10.50}$$

In practical applications, one usually specifies the expansion history $a(t)$, and then solves this equation numerically to the required precision. Here we will instead look at some approximate solutions valid for any cosmology (any $a(t)$). First, consider the superHubble regime in which the wavelength of the perturbation is much larger than the Hubble radius $k/a \ll H$. Then we can drop the spatial derivatives[109]

$$k \ll aH : \qquad \ddot{\mathcal{D}}_s + 3H\dot{\mathcal{D}}_s \simeq 0 \quad \Rightarrow \quad \dot{\mathcal{D}}_s \propto a^{-3}\,, \tag{10.51}$$

and so the two independent superHubble solutions are

$$\mathcal{D}_s(t, k \ll aH) = \left[A_s(k) + B_s(k)a^{3(w-1)/2}\right]\,, \tag{10.52}$$

The time dependent solution is decaying for $w < 1$, which is always satisfied in standard cosmologies, and so it can usually be neglected after some efoldings of superHubble evolution. In the opposite regime of subHubble perturbations $k \gg aH$, we can solve (10.45) in the WKB approximation. By making an Ansatz $\mathcal{D}_s = X(t)\exp\left[ik\int^t dt'/a(t')\right]$ and the solving the resulting differential equation for $X(t)$ to leading order in $k \gg aH$, one finds $X \propto a^{-1}$. So, the two independent subHubble solutions are

$$\mathcal{D}_s(t,k) = \frac{\tilde{A}_s\cos(k\tau(t)) + \tilde{B}_s\sin(k\tau(t))}{a} \qquad (k \gg aH)\,, \tag{10.53}$$

with $\tau \equiv \int^t dt'/a(t')$. These solutions describe the oscillations of gravitational waves as they propagate, but we also notice that the amplitudes decay as a^{-1} due to the expansion of the universe. Notice that, if parity is broken, the two polarizations plus and cross could have different initial conditions.

10.7 Scalar perturbations

It is time to tackle the most complicated and most relevant modes for cosmology: scalar perturbations. Let us start with a simple counting, assuming for simplicity that we have only one fluid[110]. We have four independent scalar equations (00, ii, longitudinal $0i$, and longitudinal ij parts of the Einstein Equations[111]) for 8 variables (four in the metric, A, B, E and F, and four in δT, namely $\delta\rho, \delta p, \delta u$ and π^S). The pressure p and anisotropic stresses $\pi^{S,V,T}$ depend on the property of the fluid under consideration and need to be specified by some constitutive equations, such as the equation of state $p = p(\rho, \dots)$. For example, for a relativistic perfect fluid $p = \rho/3$, while for a non-relativistic one $0 =\simeq p \ll \rho$. Also, for a perfect fluid, which is a good approximation in most cosmological applications, the anisotropic inertia vanish $\pi^{S,V,T} = 0$. This determines[112] two scalars, namely π^S and δp. We are still left with 6 variables for 4 equation, but we have not used the two scalar gauge transformations ϵ^0 and ϵ^S. One can now proceed in two ways. One can work only with gauge-invariant combinations, namely 4 independent linear combinations of the 6 scalars that are invariant under the gauge transformations (10.37) and (10.38). We will encounter two such variable later, (10.84). Alternatively, one can *fix the gauge* and work with a particular set of coordinates. This second approach is somewhat more convenient and will be followed in this course.

The idea of fixing the gauge is to choose coordinates that correspond to the constant hypersurfaces of some of the perturbations, so that those perturbations appear constant. In other words, we can choose ϵ^0 and ϵ^S in Eq. (10.39) in such a way to cancel whatever profile of some of the scalar perturbations, using the transformation properties in (10.37). Since there are 6 scalar perturbations but only two scalar gauge parameters, there are clearly many different possible choices (in fact infinitely many). Notice that the gauge parameters ϵ^μ need to vanish at spatial infinity in the same way as the physical perturbations they need to cancel. In this sense these are *small gauge transformations*. See below Eq. (10.87) for a discussion of large gauge transformations. Let us see the most commonly used gauge choices

Newtonian gauge Using (10.37), we see that

$$\begin{cases} \epsilon^S = a^2 B/2 \\ \epsilon_0 = aF - \frac{a^2}{2}\dot{B} \end{cases} \quad \Rightarrow \quad \begin{cases} B' = B + \Delta B = B - B = 0 \\ F' = F + \Delta F = F - F = 0\,. \end{cases} \tag{10.54}$$

[109]This is sometimes called the "separate universe" approximation because after dropping the spatial derivatives every superHubble patch of the universe evolves completely independently from the others. One can also keep subleading order in spatial derivatives.

[110]In practice there will be several different components of the universe. Some components might be interacting with each other such as electron, baryons and photons before recombination, while some components might be decoupled, such as neutrinos at $T \ll$MeV. The energy momentum tensor is separately conserved for each set of mutually interacting components.

[111]Notice that the equations for the conservation of the energy-momentum tensor are *not* independent

[112]Given a simple equation of state $p = p(\rho)$, one finds $\delta p = (\partial p/\partial\rho)\,\delta\rho$.

In a more compact form, we will simply write the gauge condition

$$B = 0 \qquad F = 0\,. \tag{10.55}$$

Notice that these two conditions determine ϵ^0 and ϵ^S completely, so small scalar gauge transformations are fully fixed by these requirements. The scalar part of the metric has then only diagonal perturbations, namely in h_{00} and h_{ii}. Traditionally these perturbations are called Φ and Ψ. So, with the identification $E = 2\Phi$ and $A = -2\Psi$, we find [113]

$$\boxed{ds^2 = -\left(1+2\Phi\right)dt^2 + a^2\left(1-2\Psi\right)dx^i\delta_{ij}dx^j}\,. \tag{10.56}$$

This is the perturbed metric in Newtonian gauge [114] . Since in this gauge $B = F = 0$, the Einstein and Energy-momentum equations simplify considerably. Because of the SVT decomposition and are gauge choice, we can find the scalar equations by substituting

$$\begin{aligned} h_{00} &= -2\Phi\,, & \delta T_{00} &= 2\bar\rho\Phi + \delta\rho\,, \\ h_{0i} &= 0\,, & \delta T_{0i} &= -\left(\bar\rho + \bar p\right)\partial_i\delta u\,, \\ h_{ij} &= -a^2\delta_{ij}2\Psi\,, & \delta T_{ij} &= a^2\partial_{ij}\pi^S + \delta_{ij}a^2\left(\delta p - \bar p 2\Psi\right)\,, \end{aligned} \tag{10.58}$$

into the linearised Einstein equations. In particular, we have four equations corresponding to the trace of (10.7) (contracting with δ^{ij}), the traceless part of (10.7), (10.8) and (10.9), which take the form

$$-\frac{1}{2M_{\text{Pl}}^2}\left[\delta\rho - \delta p - \nabla^2\pi^S\right] = H\dot\Phi + \left(4H^2 + 2\frac{\ddot a}{a}\right)\Phi - \frac{\nabla^2\Psi}{a^2} + \ddot\Psi + 6H\dot\Psi\,, \tag{10.59}$$

$$-\frac{a^2}{M_{\text{Pl}}^2}\partial_i\partial_j\pi^S = \partial_i\partial_j\left(\Phi - \Psi\right)\,, \tag{10.60}$$

$$\frac{1}{2M_{\text{Pl}}^2}\left(\bar\rho + \bar p\right)\partial_i\delta u = -H\partial_i\Phi - \partial_i\dot\Psi\,, \tag{10.61}$$

$$\frac{1}{2M_{\text{Pl}}^2}\left(\delta\rho + 3\delta p + \nabla^2\pi^S\right) = \frac{\nabla^2\Phi}{a^2} + 3H\dot\Phi + 3\ddot\Psi + 6H\dot\Psi + 6\frac{\ddot a}{a}\Phi\,. \tag{10.62}$$

The two scalar energy-momentum conservation equations ($T^{0\mu}_{\ ;\mu} = 0$ and the longitudinal part of $T^{i\mu}_{\ ;\mu} = 0$) are similarly obtained (see P.10.10)

$$\delta p + \nabla^2\pi^S + \partial_0\left[\left(\bar\rho + \bar p\right)\delta u\right] + 3H\left(\bar\rho + \bar p\right)\delta u + \left(\bar\rho + \bar p\right)\Phi = 0\,, \tag{10.63}$$

$$\delta\dot\rho + 3H\left(\delta\rho + \delta p\right) + \nabla^2\left[\frac{\left(\bar\rho + \bar p\right)}{a^2}\delta u + H\pi^S\right] - 3\left(\bar\rho + \bar p\right)\dot\Psi = 0\,. \tag{10.64}$$

A few comments are in order. First, notice also that although the energy-momentum conservation equations are not independent from the EE's, they contain one less derivative and therefore they are often more convenient to use. Second, the scalar constraint equation in this gauge is manifest in (10.60), which contains no time derivatives. In the absence of anisotropic stresses, a good approximation to our real universe, this equation is solved[115] by

$$\Phi = \Psi \quad \text{(no anisotropic inertia)}\,. \tag{10.65}$$

From the discussion around (2.24), we know that Φ corresponds to the non-relativistic Newtonian potential appearing in Newton's law of motion $\ddot x^i = -\partial^i\Phi$. Because of these two facts, both Ψ and Φ are often called *Newtonian* potentials.

[113]Be aware that this is possibly the least universal convention in physics. You might find references where the definitions of Φ and Ψ as well as their signs are exchanged. Here I follow Weinberg's notation, which differ from Dodelson's notation by $\Phi_W = \Psi_D$ and $\Psi_W = -\Phi_D$.

[114]Be aware of the existence of the closely related *conformal* Newtonian gauge, defined such that

$$ds^2 = -a^2\left[\left(1+2\Phi\right)d\tau^2 + \left(1-2\Psi\right)dx^i\delta_{ij}dx^j\right]\,. \tag{10.57}$$

[115]Notice that it is crucial to demand that Φ and Φ vanish at infinity for this solution to be unique.

Synchronous gauge* An alternative choice of gauge makes the temporal scalar part of the metric $h_{0\mu}$ vanish identically, namely one chooses ϵ^0 and ϵ^S such that

$$E = 0 \quad \text{and} \quad F = 0\,. \tag{10.66}$$

The perturbed metric takes the form

$$ds^2 = -dt^2 + a^2 dx^i dx^j \left[\delta_{ij}(1+A) + \partial_i\partial_j B\right]\,. \tag{10.67}$$

The clocks of observers at rest in these coordinates tick at the same rate, hence the name "synchronous". The four scalar Einstein equations are

$$\begin{aligned}
-\frac{1}{M_{\rm Pl}^2}\left[\delta\rho - \delta p - \nabla^2\pi^S\right] &= \frac{\nabla^2 A}{a^2} - \ddot{A} - 6H\dot{A} - H\nabla^2\dot{B} && (10.68)\\
-\frac{2}{M_{\rm Pl}^2}\partial_i\partial_j\pi^S &= \partial_i\partial_j\left[\frac{A}{a^2} - \ddot{B} - 3H\dot{B}\right]\,, && (10.69)\\
\frac{1}{M_{\rm Pl}^2}\left(\bar\rho + \bar p\right)\partial_i\delta u &= \partial_i\dot{A}\,, && (10.70)\\
-\frac{1}{M_{\rm Pl}^2}\left(\delta\rho + 3\delta p + \nabla^2\pi^S\right) &= 3\ddot{A} + 6H\dot{A} + \nabla^2\ddot{B} + 2H\nabla^2\dot{B}\,. && (10.71)
\end{aligned}$$

The two scalar energy-momentum conservation equations ($T^{0\mu}_{\ ;\mu} = 0$ and the longitudinal part of $T^{i\mu}_{\ ;\mu} = 0$) are

$$\begin{aligned}
\delta p + \nabla^2\pi^S + \partial_0\left[\left(\bar\rho + \bar p\right)\delta u\right] + 3H\left(\bar\rho + \bar p\right)\delta u &= 0\,, && (10.72)\\
\delta\dot\rho + 3H\left(\delta\rho + \delta p\right) + \nabla^2\left[\frac{\left(\bar\rho + \bar p\right)}{a^2}\delta u + H\pi^S\right] + \frac{1}{2}\left(\bar\rho + \bar p\right)\partial_0\left[3A + \nabla^2 B\right] &= 0\,. && (10.73)
\end{aligned}$$

Unlike for Newtonian gauge, the synchronous gauge conditions $E = 0 = F$ do not fix completely small gauge transformations. One can still perform a gauge transformation with

$$\epsilon_0 = -T(\mathbf{x}) \qquad \epsilon^S = a^2 T(\mathbf{x})\int\frac{dt'}{a(t')}\,, \tag{10.74}$$

which does not alter the condition $E = 0 = F$, but changes perturbations according to

$$\begin{aligned}
\Delta\Psi &= -\frac{\nabla^2 T}{a^2} - 3T\dot{H}\,, \quad \delta u = T\,, && (10.75)\\
\Delta\delta\rho &= -T\dot{\bar\rho} \quad \Delta\delta p = -T\dot{\bar p}\,. && (10.76)
\end{aligned}$$

This additional redundancy can be fixed if the universe contains a non-relativistic fluid, such as for example Dark Matter. In that case, (10.72) tells us that δu_D is constant in time (up to corrections of order $\bar p_D/\bar\rho_D \ll 1$) and one can impose the additional gauge condition $\delta u_D = 0$, which completely fixes the gauge. To transform from synchronous to Newtonian gauge we can use (see P.10.11)

$$\Phi = -\frac{1}{2}\partial_0\left(a^2 B\right)\,, \quad \Psi = -\frac{1}{2}A + \frac{a^2 H}{2}\dot{B}\,, \tag{10.77}$$

while the opposite conversion is W 5.3.46. A classic and extensive discussion of cosmological perturbation theory in Newtonian and syncronous gauges can be found in [30].

Comoving orthogonal gauge* Another option, often employed in the study of perturbation during inflation is comoving gauge[116], in which

$$\delta u = 0 \quad \text{and} \quad F = 0\,. \tag{10.78}$$

It is straightforward to check that $\delta u = 0$ fixes ϵ^0, while ϵ^S is completely fixed by the condition $F = 0$. From its definition, the linearly perturbed energy momentum tensor is (W 5.1.43)

$$\begin{aligned}
\delta T^i_j &= \delta_{ij}\delta p + \partial_{ij}\pi^S + \partial_i\pi^V_j + \partial_j\pi^V_i + \pi^T_{ij}\,, && (10.79)\\
T^i_0 = \delta T^i_0 &= \frac{\bar\rho + \bar p}{a^2}\left(a\partial_i F + aG_i - \partial_i\delta u - \delta u^V_i\right)\,, && (10.80)\\
T^0_i = \delta T^0_i &= \left(\bar\rho + \bar p\right)\left(\partial_i\delta u + \delta u^V_i\right)\,. && (10.81)
\end{aligned}$$

[116] Notice that here again there is some confusion in the literature for the use of the term comoving.

Neglecting vector modes, $G_i = 0 = \delta u_i^V$, we find that in this gauge $T_0^i = T_i^0 = 0$. The fact that $T_0^i = 0$ means that observers at rest in these coordinates are comoving with the fluid, while the fact that $T_i^0 = 0$ means that the velocity of the fluid is orthogonal to the constant time hypersurfaces. Notice that in general in this gauge $\delta\rho \neq 0$.

Constant density gauge* This is another useful, but less used gauge for inflationary perturbations. As the name suggests, one imposes

$$\delta\rho = 0 \quad \text{and} \quad F = 0\,. \tag{10.82}$$

These conditions fix the small gauge completely.

Spatially flat gauge* One last option we want to mention is to fix the spatial part of the metric to be completely unperturbed, $g_{ij} = a^2\delta_{ij}$, so that $h_{ij} = 0$. In the SVT notation one imposes

$$A = 0 \quad \text{and} \quad B = 0\,. \tag{10.83}$$

In this gauge of course $E, F \neq 0$. But these (and more generally all $h_{0\mu}$) are non-dynamical degrees of freedom, since they appear with at most first derivatives in the EE's and the initial condition $\dot{h}_{0\mu}$ cannot be specified arbitrary but is fixed by the other initial conditions. So in some sense all dynamical scalar degrees of freedom in this gauge are in the matter sector as opposed to the metric sector.

10.8 Adiabatic modes

As the reader might have painfully noticed, even at linear order and for a single fluid, the equations of motion are already quite lengthy. Things get much worse when one includes all relevant constituents of the universe and/or goes beyond linear order. In practice this is often done with numerical codes such as CLASS or CAMB (one of the first efficient and popular code was CMBFAST). These codes, often collectively referred to as *Boltzmann codes*, are routinely used in data analysis and theoretical forecasting. To solve the equations of motion one also needs initial condition. In the currently favored cosmological model, initial conditions are set up during a phase of very fast accelerated expansion in the first fraction of a second (inflation), as we mentioned in Lec. 6. One problem immediately arises when we try to evolve these initial condition forward in time since we do not know the constituents of the universe at energies much bigger than those probed at colliders, say above 10 TeV. Luckily for us, there seems to be quantities that, under certain conditions, are conserved and therefore can be trivially evolve in time. This result, which we are about to discuss, is one of the most important in cosmology. It allows one to study high energy physics by looking at the distribution of galaxies or of sub-eV photons. This remarkable connection of low-energy observables to high-energy physics has been a tremendous drive for the field of cosmology and has open new possibility to explore the fundamental laws of nature.

Let us start by introducing two new variables[117]

$$\mathcal{R} \equiv \frac{A}{2} + H\delta u\,, \tag{10.84}$$

$$\zeta \equiv \frac{A}{2} - H\frac{\delta\rho}{\dot{\bar\rho}}\,. \tag{10.85}$$

From the gauge transformations, it is straightforward to check that both $\mathcal{R}$ and ζ are gauge invariant at linear order. We will refer to $\mathcal{R}$ as *curvature perturbations on comoving hypersurfaces*, because in comoving gauge $\mathcal{R} = A/2$ and A modifies the spatial diagonal part of the metric. For the same reason, ζ is often called *curvature perturbations on constant density hypersurfaces*. The two gauge-invariant variables are related at linear order due to the equations of motion. This is most easily seen in Newtonian gauge

$$\zeta(\vec{k}, t) = \mathcal{R}(\vec{k}, t) + \frac{M_{\rm Pl}^2}{3a^2(\bar\rho + \bar p)} k^2 A(\vec{k}, t) \quad \text{(Newtonian gauge)}\,. \tag{10.86}$$

Notice that the difference $\zeta - \mathcal{R}$ is proportional to $(k/aH)^2$, and therefore is negligible outside the Hubble radius, namely for $k_{\rm phy} = k/a \ll H$. So $\mathcal{R}$ will be conserved outside the Hubble radius if ζ is, and viceversa. We are now ready to state an important theorem [48].

[117]Notice that, unfortunately, different conventions for the names of these variables exists. A useful summary of the many possible choices in the literature is given in App A of [46].

Theorem 1. *Whatever the constituents of the universe and outside the Hubble radius, $k \ll aH$, there are two conserved scalar adiabatic modes, i.e. $\dot{\mathcal{R}} = 0$, one of which satisfies $\mathcal{R} \neq 0$, and one conserved tensor mode, i.e. $\dot{D}_{ij} = 0$, for which $D_{ij} \neq 0$.*

This statement is valid to all orders in perturbation theory around a flat FLRW spacetime, but we will prove it only at linear order. Also, we will work in Newtonian gauge[118]. Consider the following large gauge transformation that maintains Newtonian gauge (see P.10.12)

$$\epsilon_\mu = \left\{\epsilon(t), a^2\omega_{ij}x^j\right\} , \tag{10.87}$$

with ϵ some time dependent by space independent function and ω_{ij} an arbitrary spacetime 3×3 constant matrix. Since ϵ^μ does not vanish at spatial infinity, its existence does not contradict the statement that Newtonian gauge conditions completely fixes the small gauge. If we start from an unperturbed flat FLRW universe, after this gauge transformation we find

$$\begin{aligned} \Phi &= -\dot{\epsilon}\,, \quad \Psi = H\epsilon - \frac{1}{3}\omega_{ii}\,, \\ \delta p &= -\dot{\bar{p}}\epsilon\,, \quad \delta\rho = -\dot{\bar{\rho}}\epsilon\,, \quad \delta u = \epsilon\,, \quad \pi^S = 0\,, \\ D_{ij} &= -\omega_{(ij)} + \frac{2}{3}\delta_{ij}\omega_{kk}\,. \end{aligned} \tag{10.88}$$

Notice that these transformation are completely different from those valid for small gauge transformations, Eq. (10.37), for example, the tensor perturbations D_{ij} is not invariant and so on. Notice that the anti-symmetric part of ω_{ij} is irrelevant since it does not generate any perturbation. Now comes the first crucial point. Since GR is a diff invariant theory and we started from unperturbed FLRW plus unperturbed $\bar{T}_{\mu\nu}$, which is a solution, the perturbations in Eq. (10.88) must be a solution of the equations of motion. This is also easily verified (see P.10.13). Recall that ϵ and ω do not vanish at spatial infinity, so this solution is an unphysical one. After all, it is just a change of coordinates.

The clever insight of Weinberg is to demand whether this gauge transformation can be extended to a physical solution. This is most easily though about in Fourier space, where the perturbations in Eq. (10.88) are all proportional do $\delta_D(\vec{k})$ and its derivative. A physical solution must eventually vanish at infinity and so its Fourier transform must be supported at $\vec{k} \neq 0$. When $\vec{k} \neq$ we are not guaranteed anymore that Eq. (10.88) is a solution. For all equations of motion that do not vanish as $\vec{\to} 0$, we know that a small modification of Eq. (10.88) is still a solution. For example, for the tensor perturbations, one can look for a solution of the form $D_{ij}(t) + \delta D_{ij}(t, \vec{k})$, where $D_{ij}(t)$ is the large perturbation in Eq. (10.88), and $\delta D_{ij}(t, \vec{k})$ is a small spatially varying (supported at $\vec{k} \neq 0$) correction. Given that we are solving linear differential equations, we can always find one such δD_{ij}. So we conclude that, whatever the constituents of the universe, there is always a solution to the equations of motion with a constant, non-vanishing D_{ij}, up corrections suppressed by k^2 in the superHubble limit. This solution represent the conservation of *primordial gravitational waves*. As we will discuss with inflation, the existence of this solution constitutes a unique opportunity to probe GR and its perturbative quantization.

The extension to a physical, non-constant solution can therefore be obstructed only when a given equation of motion vanishes identically for $\vec{k} = 0$. This happens for the off-diagonal part of the space-space Einstein equations, Eq. (10.60). We need therefore to check that this equation is solved also for $\vec{k} \neq 0$, namely

$$k_i k_j (\Phi - \Psi) = 0 \quad \Rightarrow \quad \Phi = \Psi\,. \tag{10.89}$$

This physicality condition fixes ϵ in terms of ω_{kk} as

$$\dot{\epsilon} + H\epsilon = \frac{1}{3}\omega_{kk} \quad \Rightarrow \quad \epsilon(t) = \frac{\omega_{kk}}{3a(t)}\int_T^t a(t')dt'\,, \tag{10.90}$$

where T represents some integration constant. Using this solution for ϵ and the perturbations in Eq. (10.88), we find

$$\mathcal{R} = \frac{\omega_{kk}}{3}\,. \tag{10.91}$$

[118] The theorem of can be proven in other gauges as well. In the original paper [48], Newtonian and synchronous gauges are discussed. In [11] and [24] the same derivation was presented for comoving gauge (aka "ζ-gauge") and generalized to higher order in derivatives.

We conclude that a solution with $\dot{\mathcal{R}} = 0$ and $\mathcal{R} \neq 0$ must always exist as consequence of diffeomorphism invariance. In other words, there is always a physical solution with constant $\mathcal{R}$ that sits nearby a gauge transformation. Notice that this procedure gives us the solution for metric perturbations

$$\boxed{\Phi = \Psi = \mathcal{R}\left[-1 + \frac{H}{a}\int_T^t a(t')dt'\right]}, \tag{10.92}$$

and for fluid perturbations

$$\boxed{\frac{\delta s}{\dot{\bar{s}}} = -\delta u = -\frac{\mathcal{R}}{a}\int_T^t a(t')dt'}, \tag{10.93}$$

for any diff scalar s (such as ρ and p). If we define $w = p/\rho$ for the background cosmology, these expressions give

$$\Phi = \Psi = -\mathcal{R}\frac{3(1+w)}{5+3w}. \tag{10.94}$$

$$H\frac{\delta s}{\dot{\bar{s}}} = -H\delta u = -\mathcal{R}\frac{2}{5+3w} = \Phi\frac{2}{3(1+w)}. \tag{10.95}$$

for single fluid backgrounds. These will be the initial conditions we will use to study the formation of Large Scale Structures and the Cosmic Microwave Background. Finally, in Les. 8 to Les. **??**, we will see how quantum fluctuations during inflation generate precisely these modes. Before concluding, notice that since integration constant T is arbitrary, there is actual a second solution given by the different of two solutions with different T. This solution is

$$\Phi = \Psi = \frac{CH(t)}{a(t)}, \tag{10.96}$$

$$\frac{\delta s}{\dot{\bar{s}}} = -\delta u = -\frac{\mathcal{R}}{a},. \tag{10.97}$$

and decays with time during the hot big bang.

10.9 Irreducible representations of ISO(3)*

The following discussion below paraphrases [50] Chapter 2, and I could not find an equivalent discussion in the literature). To find the irreps of ISO(3) we need to find a set of matrices $U(R,\alpha)$ for each ISO(3) element $\{R^i_j, \alpha_l\}$ that act on some Hilbert (vector) space of perturbations. In the following I will borrow the language from Quantum mechanics and refer to perturbations as "states" or "state-vectors". To begin, we note that "the component of the three-momentum all commute with each other and so it is natural to express physical state-vectors in terms of eigenvectors of the three-momentum." [50]. This is the usual Fourier transform: we consider state-vectors that are eigen-functions of translations

$$\hat{P}^i\psi_{k\sigma} = k^i\psi_{k\sigma}, \tag{10.98}$$

where σ is some other (discrete) quantum number that we have to figure out. Translations are represented by the unitary transformation

$$U(1,\alpha)\psi_{k\sigma} = e^{-ik^i\alpha_i}\psi_{k\sigma}. \tag{10.99}$$

Now, we want to find the action of rotations $U(R,0) \equiv U(R)$. Using the group properties, we note that

$$U(R)\psi_{k\sigma} = C_{\sigma\sigma'}(R,k)\psi_{Rk\sigma'}, \tag{10.100}$$

that is, a rotation changes the three-momentum of the state. We want now to find irreducible $C_{\sigma\sigma'}$ (i.e. that cannot be decomposed into smaller blocks by changing the basis for $\psi_{k\sigma}$). For this we will use the method of induced representations. The subgroup of ISO(3) we will be interested in is SO(3). The only invariant under SO(3) is the norm of a vector (and any function thereof), $k^ik^j\delta_{ij} = k^2$. Let us play some algebraic tricks now. For a reference vector q^i, define the rotation $S(k)$ that transforms it into any other vector k^i as

$$S(k)q = k \quad \Rightarrow \quad S^{-1}(k)k = q. \tag{10.101}$$

We can then re-write any state with momentum k as a transformation of a state with reference momentum q,

$$\psi_{k\sigma} = U(S(k))\psi_{q\sigma}\,. \tag{10.102}$$

Then, the action of a general rotation R can be massaged as follows:

$$\begin{aligned}
U(R)\psi_{k\sigma} &= U(R)U(S(k))\psi_{q\sigma} && (10.103)\\
&= U(S(Rk))U(S^{-1}(Rk)RS(k))\psi_{q\sigma} && (10.104)\\
&= U(S(Rk))D_{\sigma\sigma'}\psi_{q\sigma'} && (10.105)\\
&= D_{\sigma\sigma'}U(S(Rk))\psi_{q\sigma'} && (10.106)\\
&= D_{\sigma\sigma'}\psi_{Rk\sigma'}\,, && (10.107)\\
& && (10.108)
\end{aligned}$$

where in the third line we recognised that $S^{-1}(Rk)RS(k)q = q$ and so

$$U(S^{-1}(Rk)RS(k))\psi_{q\sigma} \equiv D_{\sigma\sigma'}\psi_{q\sigma}\,, \tag{10.109}$$

i.e. it must be some linear combination $D_{\sigma\sigma'}$ of states with momentum q. From this definition of $D_{\sigma\sigma'}$, we see that is it provides a representation of the little group, namely the subgroup of SO(3) that leaves the representative vector q invariant. For every little group rotation r, we have

$$U(r)\psi_{q\sigma} = D_{\sigma\sigma'}(r)\psi_{q\sigma'}\,. \tag{10.110}$$

Summarising, choosing a representative vector q and given a representation $D_{\sigma\sigma'}$ of the little group for q, we get a representation of the full group ISO(3) defined by

$$\boxed{\begin{aligned}
U(1,\alpha)\psi_{k\sigma} &= e^{-ik^i\alpha_i}\psi_{k\sigma}\,,\\
U(R,0)\psi_{k\sigma} &= D_{\sigma\sigma'}(r(R,k))\psi_{Rk\sigma'}\,,
\end{aligned}} \tag{10.111}$$

where the little group element $r(R,k)$ is given by

$$r(R,k) \equiv S^{-1}(Rk)RS(k)\,. \tag{10.112}$$

10.9.1 Little groups*

While for the Poincaré group there are 6 little groups, of which only three have physical significance (the vacuum, massive particles and massless particles), for cosmology there are only two little groups: SO(3) itself for $q^i q_i = 0$, and SO(2) for $q^i q_i \neq 0$.

The irreps of SO(3) are well known from the study of angular momentum in quantum mechanics. They are classified by the Casimir operator J^2, with eigen-values $l(l+1)$ for $l = 0, 1/2, 1, \ldots$ and are of dimension $2l+1$ with states $|l,m\rangle$ and $|m| \leq l$. Focussing on the bosonic irreps with integer l, we know they correspond to spin zero, one, two, etc. The field operators that generate those states are:

$$\begin{aligned}
&\text{Spin zero:} && \phi,\ h_{ii},\ \ldots && (10.113)\\
&\text{Spin one} && h_{0i},\ u_i,\ \ldots && (10.114)\\
&\text{Spin two:} && h_{\langle ij\rangle} \equiv h_{ij} - \frac{1}{3}h_{kk}\delta_{ij},\ \ldots\,. && (10.115)
\end{aligned}$$

Notice that the splitting between the trace of the two-tensor h_{ij}, which has spin zero, and its traceless part $h_{\langle ij\rangle}$, which has spin two, is purely algebraic and does not involve any (inverse) Laplacians. These $q = 0$ irreps are relevant to classify and discuss the background and adiabatic modes. For physical perturbations, we have to consider the other representative vector.

For $q^i q_i \neq 0$, we can choose as representative vector $q^i = \{q, 0, 0\}$ so that the little group is recognised as two-dimensional rotations, namely SO(2), which is an abelian group. All complex representations of an Abelian group are one-dimensional by Schur's lemma (all real representations are two dimensional). There are infinitely many such representations, enumerated by an integer $m \in \mathbb{N}$. Physically, we can interpret m as the "helicity" of the state, i.e. how it transforms under a rotation around the direction of its momentum. If the underlying theory is parity invariant, which is sometimes assumed in cosmological applications, for every state with helicity m there as to exist a state of helicity $-m$. So we have classify states as helicity 0, 1, 2 etc.

Problem lesson 10

P.10.1 *Otional* Find a computer with Mathematica. Install xAct and xPand following the instructions here, and use it to derive the linearised Einstein Equations in any gauge, as given in the notes.

P.10.2 Solve Eq. (10.23) for $\theta(v)$ (not $\theta(v,\omega)$). From the solution, assuming that v_i transforms as tensors under diffeomorphism (and therefore also under rotations), show explicitly that θ transforms as a scalar under rotations $\theta'(x',t)=\theta(x,t)$. Does θ transform as a scalar also under general diffs?

P.10.3 Extract all the 4 tensors, 2 transverse vectors and the transverse traceless two-tensor from the a generic symmetric two-tensor $T_{\mu\nu}$. It is sufficient to write down an appropriate number of differential equations satisfied by these objects, you do not need to write the solutions of those equations (which is anyways straightforward). To achieve this, you might want to consider acting on the tensor with various combinations of one and two spatial derivatives ∂_i.

P.10.4 Derive the gauge transformation for vectors and two-tensors Eq. (10.34) and Eq. (10.35), at linear order in ϵ^μ.

P.10.5 A change of coordinates $x'^\mu = x^\mu + \epsilon^\mu(x)$ induces a *gauge transformation* on all perturbations. In particular, the vector perturbations in the metric C_i and G_i, defined in Eq. (10.26), transform according to Eq. (10.37). Find a combination of C_i and G_i that is invariant under gauge transformations. It will help to think about the mass dimension of these two perturbations. Compare the gauge invariant combination with the equations for vectors (10.43) (see also W 5.1.50-52).

P.10.6 Compute the equation of motion for a massless scalar field, with action

$$S=\int d^4x\,\sqrt{-g}\,\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\,. \tag{10.116}$$

Compare it with the equation of motion for the tensor modes in Eq. (10.45), aka gravitational waves.

P.10.7 Assuming $\pi^T_{ij}=0$, solve the tensor equations of motion well inside and well outside the Hubble radius, $k\gg aH$ and $k\ll aH$ respectively.

P.10.8 *Optional* Compute the gauge transformations of the components of the metric A,B,C_i,D_{ij},E,F and G_i and the analogous SVT components of the energy momentum tensor. You should reproduce Eq. (10.37) and Eq. (10.38). What do you need to assume about the scaling of $\epsilon^\mu(x)$ for $\mathbf{x}\to\infty$?

P.10.9 *Optional* Verify that the actual eom's Eq. (10.41), Eq. (10.43) and Eq. (10.45) are indeed of the form Eq. (10.15). Perform a Fourier transform and check that indeed different Fourier modes decouple.

P.10.10 Derive the continuity equation in Newtonian gauge, Eq. (10.64)

P.10.11 Derive the conversion formulae from synchronous to Newtonian gauge. You should reproduce W 5.3.51-52.

P.10.12 Prove that the transformation Eq. (10.87) maintains the Newtonian gauge conditions, namely the form of the metric in Eq. (10.56). Beware that since Eq. (10.87) represents a *large* gauge transformation (it does not vanish at spatial infinity), one can still use the general gauge transformations Eq. (10.35) but *not* those in Eq. (10.37), which had been derived only for small gauge transformations, which vanish at infinity.

P.10.13 Verify that Eq. (10.88) are solutions of the Newtonian gauge equations of motion.

Check for understanding of lesson 10

cfu.10.1 What is the difference between a scalar under general diffeomorphism (a diff-scalar) and a scalar under spatial rotations (a "rotation scalar"), as discussed in this lesson? Given an example of a diff-scalar, a rotation scalar that is also a diff-scalar and a rotation scalar that is not a diff scalar.

cfu.10.2 In this lesson, we saw that the isometries of the background suggest a way to organize perturbation theory that greatly simplifies the algebraic complexity of the equations (SVT and Fourier decomposition). Give at least three more examples of such a simplification in a classical theory, in quantum mechanics and in general relativity.

cfu.10.3 Write down a simple example to show that different Fourier modes and different helicities couple to each other at second order.

cfu.10.4 What happens to different Fourier modes and different helicities at linear order if the cosmological background depends on some spatial coordinate and spatial translations are broken? Write down a schematic simple example

cfu.10.5 Write down the form of all possible terms that are allowed by symmetry and general considerations to appear in the eom for tensor modes D_{ij}. Compare this general expectation with the actual equation Eq. (10.45).

LESSON A

A toolkit to study an equation

In every subject there are a few pivotal equations that needs to be understood as well as possible. Here I collect a step-by-step toolkit to study a given equation for the first time, with the goal of understanding its many implications. A partial, semi-ordered list of things to do contains:

1. **Form** Stare at the equation as you would stare at a beautiful painting. Take at least 30 seconds to just look at it. Discover all of its tiny indices, hidden dependences, overall form. Is it an algebraic or differential equation? If differential, to what order? Is it partial or ordinary?
2. **Variables** Enumerate and characterize the variables in the equations: what are they functions of, how do they appear (e.g. with or without derivatives, integrated over, implicitly, ...)
3. **Dimensional analysis** Know/review the mass dimension (or other dimension is $\hbar \neq 1 \neq c$) of every single parameter, variable and function appearing in the equation. Be sure to master this.
4. **Symmetries** Discuss the symmetries of the equation: is it covariant (i.e. invariant in form) under change of coordinates? is it exactly/approximately invariant under some other symmetry? How do you build new solutions from known ones?
5. **Limits** Enumerate simple limits in which the equation takes a simple, well-known or intuitive form or in which you know a (simple) solution

As an example, let me discuss the geodesic equation,

$$\boxed{\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{du}\frac{dx^\beta}{du} = 0}\,. \tag{A.1}$$

1. **Form** Four second order partial differential equations for four variables $x^\mu(u)$, with two terms and Lorentz indices. Γ is evaluated at x^μ, and therefore depends implicitly on it.
2. **Variables** The particle spacetime position $x^\mu(u)$ as function of *proper* time u (or an affine transformation thereof $u' = \lambda u + c$). Γ are the Christauwful symbols, related to the metric and its first derivative as in Eq. (2.12). x^μ appears explicitly only with (time) derivatives (one or two derivatives), but it may appear without derivative inside Γ, if e.g. the metric is not translation invariant. The metric appears both without derivatives and with one derivative. In typical applications, the metric determines the “background” and it is not a “dynamical” variable in this equation.
3. **Dimensional analysis** $[x^\mu, u] = M^{-1}$, $[\Gamma] = M^1$, $[g_{\mu\nu}, g^{\mu\nu}] = M^0$. Each term in the equation is an overall M^{-1}.
4. **Symmetries** The full equation is covariant under general spacetime diffs. Only dx^μ/du is covariant, while d^2x^μ/du^2 and Γ are not. The two terms are not separately covariant. u is proper time and therefore invariant under diffs. The theory is *not* invariant under a general reparameterization of the particle worldline $u' = u'(u)$.
5. **Limits** In the local inertial frame (which always exists thanks to the equivalence principle), the geodesic equation becomes simply $\ddot{a} = 0$. In the Newtonian limit (see Les. **??**), one finds $\ddot{a}_i = -\partial_i\phi$, as it should be.

LESSON B

Lesson references and further reading

Cosmology There are many good introductory textbooks to cosmology. I especially like those by Scott Dodelson [13], Viatcheslav Mukhanov [35] and Steven Weinberg [51]. Where possible I follow Weinberg's notation.

Sec. 2 This Lesson follows App. B of Weinberg's book, Sections 2.1, 2.3 of Dodelson and selected topics from Blau's notes and Carrol's book.

Lec. 3 The discussion of isometries and FLRW spacetime follows Ch. 13 of Weinberg's old book []. The rest is very standard.

Lec. 4 The discussion of distances follows 2.2 of Dodelson. Curvature is discussed following 1.3.1 of Mukhanov.

Lec. 5 Further details can be found in specialized reviews: for Dark Energy see [9, 43]; for neutrinos see [14, 22, 26, 27]; for Dark Matter see [5].

Lec. 6 Thermal history is summarized in most textbook, see e.g. Mukhanov 3 and expecially 3.2. The discussion of the Boltzmann equation follows closely 3.1 of Dodelson.

Lec. 7 In the Part III course I cover only BBN, but I leave here some material on recombination and Dark Matter decoupling. They all follow closely 4 of Dodelson.

Sec. 8 The horizon and flatness problems can be found in any textbook. The discussion of coherent superHubble perturbations was inspired by [12], while that of scale invariance borrows from **??** and **??**.

Sec. 8.3 A nice introductory discussion of dS and conformal diagram is given in Sec 1.3.6 and Sec. 2.3 of [35]. A more advanced discussion including QFT and Quantum Gravity in dS can be found in [45].

Sec. 9 The general discussion of inflation and slow-roll parameters can be found in any textbooks.

Les. 10 and Les. 10.7 In these lecture notes I have mostly followed Weinberg's book [51]. The equivalent chapter in Dodelson's book is Ch. 5. Two classic references on Cosmological Perturbation Theory are the review by Sasaki and Kodama [25] and that by Mukhanov, Feldman and Brandenberger [36].

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