Field Theory in Cosmology: Example Sheet 1

1. For a $P(X,\phi)$ theory

$$S = \int \sqrt{-g} P(X,\phi) \,, \tag{1}$$

compute the equations of motion. Compute the energy-momentum tensor and find the identification upon which it reduces to that of a perfect fluid

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + g_{\mu\nu} p \,. \tag{2}$$

Re-derive the equations of motion by combining the two Friedmann equations, which for a perfect fluid take the general form

$$3M_{\mathrm{P}l}^2 H^2 = \rho \,, \qquad -\dot{H} M_{\mathrm{P}l}^2 = \frac{1}{2}\left(\rho + p\right) \,. \tag{3}$$

2. Compute the power spectrum of a massive scalar field in de Sitter. Consider the action

$$S = -\int \sqrt{-g} \frac{1}{2}\left[\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right] \,, \tag{4}$$

for some mass m. Write $\phi(\mathbf{k})$ in terms of creation and annihilation operators $\{a_{\mathbf{k}}, a^\dagger_{\mathbf{k}}\}$ and mode functions f_k. Derive the equation that $f_k(\tau)$ has to satisfy from the action (4), using conformal time. To solve this equation, re-write it as an equation for $g_k = (-\tau)^{-3/2} f_k$, and then use the fact that the two linear independent solution of Bessel's differential equation ,

$$x^2 \partial_x^2 y + x \partial_x y + (x^2 - \alpha^2) y = 0 \,, \tag{5}$$

can be taken to be the two Hankel functions $H_\alpha^{(1,2)}$. Now that you have the most general solution for f_k, with two integration constant, match this solution in the $-k\tau \to \infty$ limit to the flat space solution. You may use the following expansions of the Hankel functions for $x \to \infty$

$$H_\alpha^{(1)}(x) \simeq \sqrt{\frac{2}{\pi}} \frac{e^{ix}}{\sqrt{x}} \,, \qquad H_\alpha^{(2)}(x) \simeq \sqrt{\frac{2}{\pi}} \frac{e^{-ix}}{\sqrt{x}} \,, \tag{6}$$

which are valid up to an irrelevant (α-dependent) phase. You should find

$$f_k(\tau) = \frac{\sqrt{\pi} H}{2} (-\tau)^{3/2} H_\nu^{(1)}(-k\tau) \,, \quad \nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \,. \tag{7}$$

3. Compute the two-point correlators

$$\lim_{\tau\to 0} \langle \phi(\mathbf{k}) \pi(\mathbf{k}') \rangle = (2\pi)^3 \delta_D^3 \left(\mathbf{k} + \mathbf{k}'\right) \frac{1}{2k\tau} \,, \tag{8}$$

$$\lim_{\tau\to 0} \langle \phi(\mathbf{k}) \pi(\mathbf{k}') \rangle = \lim_{\tau\to 0} \langle \pi(\mathbf{k}) \phi(\mathbf{k}') \rangle \,, \tag{9}$$

$$\lim_{\tau\to 0} \langle \pi(\mathbf{k}) \pi(\mathbf{k}') \rangle = (2\pi)^3 \delta_D^3 \left(\mathbf{k} + \mathbf{k}'\right) \frac{k}{2H^2\tau^2} \,. \tag{10}$$

4. Using the power spectrum derived in the lecture, compute the (real space) correlation function at separate points for a massless scalar field in dS and show that it is IR divergent:

$$\langle \phi(\mathbf{x}) \phi(\mathbf{0}) \rangle = \frac{H^2}{(2\pi)^2} \int_0^\infty d\tilde{k}\, \frac{\sin \tilde{k}}{\tilde{k}^2} \,. \tag{11}$$

5. Compute the amount of particle production in dS. In the lectures, we fixed the mode functions by demanding that φ creates positive-energy particles at $k\tau \to -\infty$. Let's instead require that φ creates positive-energy particles at some finite $|\tau_*| > \infty$, still satisfying $|k\tau_*| \gg 1$. The quantized field then takes the form

$$\varphi(\mathbf{k}) = g_k b_{\mathbf{k}} + g_k^* b_{-\mathbf{k}}^\dagger \,, \tag{12}$$

where $\{b_{\mathbf{k}}, b_{\mathbf{k}}^\dagger\}$ are a new set of creation and annihilation operators. Define the new vacuum state $|\tilde{0}\rangle$. Find g_k by matching to the Minkowski vacuum at τ_* (you may multiply g_k by a convenient phase)

$$g_k = \frac{H}{\sqrt{2k^3}}\left[1 + \frac{i}{k\tau_*} - \frac{1}{2(k\tau_*)^2}\right] f_k(\tau) + e^{-2ik\tau_*}\frac{H}{\sqrt{2k^3}}\frac{1}{2(k\tau_*)^2} f_k^*(\tau) \,,$$

By matching (12) to the expressions for $\varphi(\mathbf{k})$ we found in the lectures (i.e. matching to Minkowski at $|\tau_*| \to \infty$), show that the two sets of ladder operators are related,

$$a_{\mathbf{k}} = \frac{\sqrt{2k^3}}{H}\left(\alpha b_{\mathbf{k}} + \beta^* b_{-\mathbf{k}}^\dagger\right) , \qquad a_{\mathbf{k}}^\dagger = \frac{\sqrt{2k^3}}{H}\left(\beta b_{-\mathbf{k}} + \alpha^* b_{\mathbf{k}}^\dagger\right) , \tag{13}$$

This relation is called a *Bogoliubov transformation.* Invert it to give

$$b_{\mathbf{k}} = \frac{\sqrt{2k^3}}{H}\left(\alpha^* a_{\mathbf{k}} + \beta^* a_{-\mathbf{k}}^\dagger\right) , \qquad b_{\mathbf{k}}^\dagger = \frac{\sqrt{2k^3}}{H}\left(\beta a_{-\mathbf{k}} + \alpha a_{\mathbf{k}}^\dagger\right) , \tag{14}$$

Now we want to ask what a detector that measures $b_k^\dagger$ excitations would measure in the Bunch Davies vacuum $|0\rangle$, which we defined in the lecture as $a_{\mathbf{k}}|0\rangle = 0$. To this end, let's define the "b-particle" number operator

$$N_b(\mathbf{k}) = b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \,. \tag{15}$$

Compute the expectation value of $N_b(\mathbf{k})$ on the state $|\tilde{0}\rangle$ and on the Bunch-Davies vacuum $|0\rangle$. To understand the singular factor $\delta_D^3(\mathbf{0})$, work at finite volume

$$(2\pi)^3\delta_D^3(\mathbf{0}) = \lim_{V\to\infty}\int_V d^3x e^{-i\mathbf{0}\cdot\mathbf{x}} = \lim_{V\to\infty} V \,, \tag{16}$$

and define the number density of particles, $n_b(\mathbf{k}) \equiv N_b(\mathbf{k})/V$, instead of the total number $N_b(\mathbf{k})$. You should find that the Bunch-Davies state has a non-vanishing density of b-type particles given by

$$\langle 0| n_b(\mathbf{k}) |0\rangle = \frac{1}{4(k\tau)^4} \neq 0 \,. \tag{17}$$

6. The fact that an FLRW background is invariant under translations, $\mathbf{x} \to \mathbf{x} + \mathbf{b}$, implies that also correlators must be invariant

$$\langle \phi(\mathbf{x}_1)\dots\phi(\mathbf{x}_n)\rangle = \langle \phi(\mathbf{x}_1 + \mathbf{b})\dots\phi(\mathbf{x}_n + \mathbf{b})\rangle \,. \tag{18}$$

Using this, prove that momentum space correlators must always be proportional to a delta function of the total momentum

$$\langle \phi(\mathbf{k}_1)\dots\phi(\mathbf{k}_n)\rangle \propto \delta_D^3\left(\sum_{a=1}^n \mathbf{k}_a\right) \,. \tag{19}$$

7. For the metric

$$ds^2 = -dt^2 + a^2\left(\delta_{ij} + \gamma_{ij}\right) dx^i dx^j \tag{20}$$

$$= \frac{1}{H^2\tau^2}\left[-d\tau^2 + \left(\delta_{ij} + \gamma_{ij}\right) dx^i dx^j\right] , \tag{21}$$

where $\gamma_{ii} = \partial_i \gamma_{ij} = 0$, we want to expand the Einstein-Hilbert action in de Sitter

$$S_2 = \int d^4x \sqrt{-g} \left[\frac{M_{\rm Pl}^2}{2} R - \Lambda \right] \tag{22}$$

to second order in γ to find the action of a free graviton. You already performed a similar expansion around Minkowski in the General Relativity course and it was a painful calculation. Instead of doing it again, let's use a trick. Start by noticing that the dS metric is proportional to the Minkowski one

$$g_{\mu\nu}^{\rm dS} = a^2 g_{\mu\nu}^{\rm Mink} = \frac{1}{H^2\tau^2} g_{\mu\nu}^{\rm Mink} \,, \tag{23}$$

with the identification $\tau^{(\rm dS)} = t^{(\rm Mink)}$. Notice that by the Friedmann equation

$$3M_{\rm Pl}^2 H^2 = \Lambda \tag{24}$$

A metric with this property is called *conformally flat.* Given an arbitrary function Ω of the coordinates, the rescaling

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \,, \tag{25}$$

is called a Weyl transformation. Various GR tensors transform quite easily under a Weyl rescaling. For example, the Ricci scalars $\tilde{R} \equiv \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu}$ and $R \equiv g^{\mu\nu}R_{\mu\nu}$ for the two metrics are related by [this can be proven by direct calculation, if you wish]

$$\tilde{R} = \Omega^{-2} \left[R - 6\nabla_\mu \nabla^\mu \ln\Omega - 6(\nabla_\mu \ln\Omega)(\nabla^\mu \ln\Omega) \right] . \tag{26}$$

Now recall that in Minkowski, you found

$$S_2^{\rm Mink} = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g} R \tag{27}$$

$$= \frac{M_{\rm Pl}^2}{8} \int d^3x dt \left[\dot{\gamma}_{ij}\dot{\gamma}_{ij} - \partial_k\gamma_{ij}\partial_k\gamma_{ij} \right] + \mathcal{O}(\gamma^3) \quad \text{(Minkowski)} \tag{28}$$

Use (26) to rewrite the Einstein-Hilbert action around dS in terms of that around Minkowski, for which you can use the expansion above. You should find that around dS the graviton free action is

$$S_2 = \frac{M_{\rm Pl}^2}{8} \int d^3x d\tau \, a^2 \left[\gamma'_{ij}\gamma'_{ij} - \partial_k\gamma_{ij}\partial_k\gamma_{ij} \right] . \tag{29}$$

8. Prove that the two in-in expressions for a generic in-in correlator

$$\langle \mathcal{O}(t) \rangle = \sum_{N=0}^{\infty} i^N \int_{-\infty}^{t} dt_N \int_{-\infty}^{t_N} dt_{N-1} \dots \int_{-\infty}^{t_2} dt_1 \tag{30}$$
$$\times \langle 0 | [\hat{H}_{int}(t_1), [\hat{H}_{int}(t_2), \dots [\hat{H}_{int}(t_N), \mathcal{O}(t)] \dots]] | 0 \rangle \,,$$

$$\langle \mathcal{O}(t) \rangle = \langle 0 | \left[\bar{T} e^{\left(i \int_{-\infty(1+i\epsilon)}^{t} dt' \hat{H}_{int}(t') \right)} \right] \mathcal{O}(t) \left[T e^{\left(-i \int_{-\infty(1-i\epsilon)}^{t} dt' \hat{H}_{int}(t') \right)} \right] | 0 \rangle \,, \tag{31}$$

are indeed equivalent. Proceed by induction. First prove that they are equivalent at order $N = 0$ and $N = 1$. Then, assuming that they agree at order $N-1$, take the time derive of each Nth-order expression and rewrite it as the correlators of some other field to order $N-1$. This proves that the expression agree to order N up to a constant. By taking the limit $t \to -\infty$ show that the constant has to vanish.

9. Using the in-in formalism, compute the bispectrum in a $P(X)$ theory induced by the interactions $\dot{\varphi}^3$ and $\dot{\varphi}(\partial_i\varphi)^2$.

10. The fact that the de Sitter metric,

$$ds^2 = \frac{-d\tau^2 + dx^i dx^j \delta_{ij}}{\tau^2 H^2} \,, \tag{32}$$

is invariant under dilations, $\{\tau, \mathbf{x}\} \to \lambda\{\tau, \mathbf{x}\}$, implies that equal time correlators that do not depend on time, such as for example the power spectrum of a massless scalar field or of the graviton at $\tau \to 0$, must obey

$$\langle \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\ldots\phi(\mathbf{x}_n)\rangle = \langle \phi(\lambda\mathbf{x}_1)\phi(\lambda\mathbf{x}_2)\ldots\phi(\lambda\mathbf{x}_3)\rangle \,. \tag{33}$$

Using this, prove that momentum space correlators B_n, defined as

$$\langle \phi(\mathbf{k}_1)\ldots\phi(\mathbf{k}_n)\rangle = (2\pi)^3\delta_D^3\left(\sum_{a=1}^{n}\mathbf{k}_a\right) B_n(\mathbf{k}_1,\ldots,\mathbf{k}_n)\,. \tag{34}$$

must scale as

$$B_n(\lambda\mathbf{k}_1,\ldots,\lambda\mathbf{k}_n) = \frac{1}{\lambda^{3(n-1)}}B_n(\mathbf{k}_1,\ldots,\mathbf{k}_n)\,. \tag{35}$$