Field Theory in Cosmology: Example Sheet 2

1. Reproduce the constraint equations by varying the action

$$S=\int d^4x\sqrt{h}N\left\{\frac{M_{\rm Pl}^2}{2}\left[{}^{(3)}R+K_{ij}K^{ij}-K^2\right]+P(X,\phi)\right\}\,. \tag{1}$$

   with respect to N and N^i.

2. Derive the linear-order gauge transformations of A, B, ψ and h_{00}, for a generic change of coordinates $x^\mu \to x^\mu+\epsilon^\mu$.

3. Derive the gauge transformation from Newtonian gauge to flat gauge and viceversa. In particular, given some generic perturbations $\{A^N, h_{00}^N, \varphi^N\}$ in Newtonian gauge, determine teh corresponding perturbations $\{h_{00}^f, \psi^f, \varphi^f\}$ in flat gauge.

4. Solve the $\delta S/\delta N^i$ constraint to find δN, working in flat gauge to linear order.

5. In the lecture, we prove the conservation of $\mathcal{R}$ and γ_{ij} on superHubble scales in the presence of a generic energy-momentum tensor $T_{\mu\nu}$ by working in comoving gauge. Prove again the conservation of $\mathcal{R}$ by working in Newtonian gauge. In particular, you might want to start with the change of coordinates

$$\epsilon^\mu=\left\{\epsilon(t),\lambda x^i\right\}\,. \tag{2}$$

   and show that the gauge transformations are

$$\begin{aligned}
&\Phi=-\dot\epsilon\,, && \Psi=H\epsilon-\frac{\lambda}{3}\,. && \\
&\delta\rho=-\dot{\bar\rho}\epsilon\,, && \delta u=\epsilon\,, && \pi^S=0\,, \qquad (3)\\
&\delta p=-\dot{\bar p}\epsilon\,, && \varphi=-\epsilon\dot{\bar\phi}\,. && \qquad (4)
\end{aligned}$$

   Then use the scalar part of the ij components of the Einstein's equation,

$$k_ik_j\left(\Phi-\Psi\right)=0, \tag{5}$$

   to impose the physicality condition on $\epsilon(t)$. Your final result should be

$$\mathcal{R}=\frac{\lambda}{3}\,,\qquad \varphi=-\dot{\bar\phi}\frac{\mathcal{R}}{a}\int_T^t a(t')dt'\,,\qquad \Phi=\Psi=\mathcal{R}\left[-1+\frac{H}{a}\int_T^t a(t')dt'\right]\,. \tag{6}$$

6.