Stability of a general class of distributed power control algorithms for time-varying wireless networks

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Outline

1. Formulation of the problem
2. Undelayed system
3. Delayed system
4. Asymptotic bounds
5. Example
6. Summary


The Foschini–Miljanic Algorithm

\[ \frac{dp_i(t)}{dt} = k_i \left( -p_i(t) + \frac{\gamma_i}{G_{ii}} \left( \sum_{j \neq i} G_{ij} p_j(t) + \nu_i \right) \right) \]

\( p_i(t) = \) power transmitted by user \( i \).

\( k_i = \) positive scaling constant.

\( \gamma_i = \) target signal-to-interference ratio; the desired value of

\[ \frac{p_i}{\frac{1}{G_{ii}} \left( \sum_{j \neq i} G_{ij} p_j + \nu_i \right)} , \]

\( i \) and \( j \) take values in \( \{1, 2, \ldots, N\} \).

Converges to equilibrium \( p^* \) satisfying

\[ p^*_i = \gamma_i \frac{1}{G_{ii}} \left( \sum_{j \neq i} G_{ij} p^*_j + \nu_i \right) , \]

if such a power distribution exists [Foschini, Miljanic 1993].
Generalization

Idea: replace interference term \( \frac{\gamma_i}{G_{ii}} \left( \sum_{j \neq i} G_{ij} p_j + \nu_i \right) \) by a general nonlinearity satisfying certain properties. Studied in:

- discrete-time setting in [Yates 1995],

- continuous-time setting in [Lestas 2009].

Both assumed that the system nonlinearity is time-independent. What if we allow the nonlinearity to have explicit time dependence?
So consider the system:

\[
\frac{dp_i(t)}{dt} = k_i (-p_i(t) + l_i(t, p)),
\]

with \(l\) satisfying:

1. **Monotonicity**: if \(p \geq p'\), then \(l(t, p) \geq l(t, p')\),
2. **Scalability**: there exists a continuous function \(\delta : (1, \infty) \to \mathbb{R}^+\) such that, for any \(\alpha > 1\), \(l_i(t, p) - \frac{1}{\alpha} l_i(t, \alpha p) \geq \delta(\alpha)\) for all \(i \in \{1, 2, \ldots, N\}\),

for all \(t \geq 0, p \geq 0\) component-wise, and \(i \in \{1, 2, \ldots, N\}\).
Useful properties

Two important lemmas follow quickly from the properties of $I$, namely, at all times $t$:

**Lemma (continuity):** $I(t, p)$ is continuous in $p$ for all $p \geq 0$.

**Lemma (positivity):** There exists a constant $c > 0$ (independent of $t$) such that $I(t, p) > c$ for all $p \geq 0$. 
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Theorem

Suppose that $p = P(t)$ is a bounded solution of our system subject to the initial condition $P(0) > 0$. Then $P(t)$ is a uniformly asymptotically stable trajectory.

Proof:

**Step 1:** Positivity $\implies$ if $p(\tau) < c$, then $\frac{dp_i(\tau)}{dt} > 0$

$\implies p(t) \geq \min\{c, \min_i p_i(0)\} > 0$ for all $t$.

$\therefore \pi_i = \log \left( \frac{p_i}{P_i} \right)$ transforms our system to

$$\frac{d\pi_i}{dt} = \frac{k_i}{P_i e^{\pi_i}} \left[ l_i(t, \text{diag}(e^{\pi_j})P) - e^{\pi_i} l_i(t, P) \right],$$

with $p = P(t)$ transformed to an equilibrium at $\pi = 0$. 
Step 2: Let $V(\pi) = \max_i |\pi_i| \overset{\text{def}}{=} |\pi_{im}|$. Continuous + positive definite, with derivative

$$
\frac{d}{dt} V(\pi) = \begin{cases} 
\dot{\pi}_{im}(t)(t) & \text{if } \pi_{im}(t)(t) > 0, \\
-\dot{\pi}_{im}(t)(t) & \text{if } \pi_{im}(t)(t) < 0, \\
0 & \text{if } \pi_{im}(t)(t) = 0.
\end{cases}
$$

(a) If $\pi_i > 0$ and $\pi_i \geq \pi_j$ for all $j$:

$$
e^{\pi_i} l_i(t, P) > l_i(t, e^{\pi_i} P) + e^{\pi_i} \delta(e^{\pi_i}) \\
\geq l_i(t, \text{diag}(e^{\pi_j} P)) + e^{\pi_i} \delta(e^{\pi_i}),
$$

$\Rightarrow \dot{\pi}_i < -\frac{k_i}{P_i} \delta(e^{\pi_i}).$

(b) $\pi_i < 0$ and $\pi_i \leq \pi_j$ for all $j$

$$
l_i(t, \text{diag}(e^{\pi_j} P)) \\
> e^{\pi_i} l_i(t, e^{-\pi_i} \text{diag}(e^{\pi_j} P)) + \delta(e^{-\pi_i}) \\
\geq e^{\pi_i} l_i(t, P) + e^{\pi_i} \delta(e^{-\pi_i}),
$$

$\Rightarrow \dot{\pi}_i > \frac{k_i}{P_i} \delta(e^{-\pi_i}).$
Let $B$ be an upper bound for $P$ and $\kappa = \min_i k_i$.

$$\therefore \frac{d}{dt} V(\pi) \begin{cases} < -\frac{\kappa \delta(e^{V(\pi)})}{B} & \text{if some } \pi_j \neq 0, \\ = 0 & \text{if } \pi = 0. \end{cases}$$

The right-hand side is a continuous, negative-definite function of $\pi$ alone. Therefore, $p = P(t)$ is uniformly asymptotically stable. $\square$
In fact, $V(\pi) = \max_i |\pi_i|$ is radially unbounded, so $\pi = 0$ is a globally uniformly asymptotically stable equilibrium. Therefore, every trajectory $p(t)$ with $p(0) > 0$ will tend uniformly to $P(t)$ as $t \to \infty$.

Moreover, the positivity lemma means that $p_i(0) = 0 \Rightarrow p_i(\epsilon) > 0$ for any $\epsilon > 0$. Thus, start at any positive initial time to incorporate trajectories with some components of initial data equal to zero.

In this way, we obtain the final result.

**Theorem**

*Suppose that $p = P(t)$ is a bounded solution of our system subject to the initial condition $P(0) \geq 0$. Then any trajectory $p(t)$ with $p(0) \geq 0$ will tend uniformly to $P(t)$ as $t \to \infty$.***
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Incorporating time delays

Now work with the system

\[ \frac{dp_i(t)}{dt} = k_i \left( -p_i(t) + l_i(t, p^{d_i}(t)) \right), \]

where

\[ p^{d_i}(t) = (p_1(t - \theta_{i1}(t)), p_2(t - \theta_{i2}(t)), \ldots, p_N(t - \theta_{iN}(t)))^T, \]

with all delays $\theta_{ij}$ restricted to lie in some interval $[0, r]$.

Strengthen the scalability assumption to require that $\delta$ be non-decreasing.
How to deal with such systems

Theorem (Razumikhin Theorem)

Let $x = 0$ be a solution of $\dot{x}_i = f_i(t, x^{d_i})$. Suppose that $q, u, v, w : \bar{R}_+ \to \bar{R}_+$ are continuous non-decreasing functions with $q(s) > s$ and $u(s), v(s), w(s) > 0$ for all $s > 0$, $u(0) = v(0) = 0$, and $v$ strictly increasing. Suppose further that there exists a continuous function $V : \bar{R}_+ \times \mathbb{R}^N \to \mathbb{R}$ such that:

1. $u(|x|) \leq V(t, x) \leq v(|x|), \forall t \in \bar{R}_+, \forall x \in \mathbb{R}^N$,
2. $\dot{V}(t, x(t)) \leq -w(|x(t)|)$ if $V(t + \theta, x(t + \theta)) \leq q(V(t, x(t)))$ for all $\theta \in [-r, 0]$, where $x(t)$ is any trajectory.

Then the solution $x = 0$ is uniformly asymptotically stable. If moreover $u(s) \to \infty$ as $s \to \infty$, then $x = 0$ is globally uniformly asymptotically stable.
Theorem

Suppose that \( p = P(t) \) is a solution of our delayed system satisfying \( P(\theta) > 0 \) for all \( \theta \in [-r, 0] \). Suppose further that there exists a positive constant \( C \) such that \( I_i(t, P^d_i(t)) < C \) for all \( i \in \{1, 2, \ldots, N\} \) and all \( t \geq 0 \). Then \( P(t) \) is uniformly asymptotically stable.

Proof:

Step 1: As before, transform as \( \pi_i = \log \left( \frac{p_i}{P_i} \right) \). This gives

\[
\frac{d\pi_i}{dt} = \frac{k_i}{P_i e^{\pi_i}} \left[ I_i(t, \text{diag}(e^{\pi_j} P^d_i)) - e^{\pi_i} I_i(t, P^d_i) \right]
\]

and maps \( p = P(t) \) to \( \pi = 0 \).
Step 2: Again, consider $V(\pi) = \max_i |\pi_i| \overset{\text{def}}{=} |\pi_{im}|$. Continuity + positive definiteness account for condition 1 in the Razumikhin Theorem. For condition 2:

$$
\frac{dV}{dt} = \begin{cases} 
\dot{\pi}_{im}(t)(t) & \text{if } \pi_{im}(t)(t) > 0, \\
-\dot{\pi}_{im}(t)(t) & \text{if } \pi_{im}(t)(t) < 0, \\
0 & \text{if } \pi_{im}(t)(t + \theta) = 0, \forall \theta \in [-r, 0].
\end{cases}
$$

Write $q(s) = s + f(s)$, and suppose

$$
q(V(t, \pi(t))) \geq \sup_{-r \leq \theta \leq 0} V(t + \theta, \pi(t + \theta)).
$$

∴ $q(|\pi_{im}(t)(t)|) \geq \sup_{-r \leq \theta \leq 0} |\pi_{im}(t+\theta)(t + \theta)| \geq |\pi_{im}(t)|$. 
Write $i$ for $i_m$. Two significant cases:

(a) $\pi_i > 0$, whence $\pi^d_j - q(\pi_i) \leq 0$ and

$$l_i \left(t, \text{diag} \left( e^{\pi^d_j} \right) P^d_i \right) = l_i \left(t, e^{q(\pi_i)} \text{diag} \left( e^{\pi^d_j - q(\pi_i)} \right) P^d_i \right)$$

$$\leq l_i \left(t, e^{q(\pi_i)} P^d_i \right)$$

$$\leq e^{q(\pi_i)} \left( l_i \left(t, P^d_i \right) - \delta \left(e^{q(\pi_i)}\right)\right)$$

$$\leq e^{q(\pi_i)} \left( l_i \left(t, P^d_i \right) - \delta(\pi_i)\right).$$

Now suppose that $f \leq f_1$ for some continuous function $f_1$ satisfying $f_1(0) = 0$ and $f_1(s) > 0$ for all $s > 0$. 
\[ \dot{\pi}_i \leq \frac{k_i}{P_i} \left[ l_i \left( t, P_d^i \right) \left( e^{f_1(\pi_i)} - 1 \right) - e^{f_1(\pi_i)} \delta(e^{\pi_i}) \right] \]
\[ < \frac{k_i}{P_i} \left[ C \left( e^{f_1(\pi_i)} - 1 \right) - \delta(e^{\pi_i}) \right]. \]

Upper bound on \( l_i(t, P_d^i) \) gives upper bound on \( P_i(t) \) - call this \( B \). Then we get
\[ \dot{\pi}_i < -\frac{\kappa \delta(e^{\pi_i})}{2B}, \]
provided that
\[ 1 < e^{f_1(\pi_i)} < 1 + \frac{\delta(e^{\pi_i})}{2C}. \]

Such an \( f_1 \) can be chosen to be non-decreasing.
(b) $\pi_i < 0$. Analogous arguments yield

$$\dot{\pi}_i > \frac{\kappa \delta (e^{-\pi_i})}{2B},$$

when $f \leq f_2$, $f_2(0) = 0$, $f_2(s) > 0$ for all $s > 0$, and

$$1 - \frac{\delta (e^{-\pi_i})}{2C} < e^{-f_2(-\pi_i)} < 1.$$

Let $f(s) = \min\{f_1(s), f_2(s)\}$ and $w(s) = \frac{\kappa \delta (e^s)}{2B}$, and then $q(s) = s + f(s)$ and $w(s)$ will satisfy all the required properties for condition 2 to hold.
Just as in the undelayed case, it is straightforward to extend this theorem to the following, stronger result.

**Theorem**

Suppose that $p = P(t)$ is a solution of our delayed system satisfying $P(\theta) \geq 0$ for all $\theta \in [-r, 0]$. Suppose further that there exists a positive constant $C$ such that $l_i(t, P^{d_i}(t)) < C$ for all $i \in \{1, 2, \ldots, N\}$ and all $t \geq 0$. Then any trajectory $p(t)$ with $p(\theta) \geq 0$ for $\theta \in [-r, 0]$ will tend uniformly to $P(t)$ as $t \to \infty$. 
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Theorem (An auxiliary result)

Suppose:

- $J^{\text{min}}(p)$ and $J^{\text{max}}(p)$ satisfying monotonicity and scalability,
- $J^{\text{min}}(p) \leq J^{\text{max}}(p)$ for all $p \geq 0$,
- $\frac{dp_i(t)}{dt} = k_i (-p_i(t) + J_i(p))$ for $J = J^{\text{min}}$ and $J = J^{\text{max}}$ have equilibria $q^{\text{min}} \geq 0$ and $q^{\text{max}} \geq 0$ respectively,
- $J^{\text{min}}(q^{\text{min}}) < I(t, q^{\text{min}})$ and $I(t, q^{\text{max}}) < J^{\text{max}}(q^{\text{max}})$ for all $t \geq 0$.

Then the set

$$D = \{ p : q^{\text{min}} \leq p \leq q^{\text{max}} \}.$$

is positively invariant with respect to our delayed system, in the sense that if $p(\theta) \in D$ for all $\theta \in [-r, 0]$, then $p(t) \in D$ for all $t \geq 0$.

In particular, the foregoing results (as appropriate) apply.
Delayed Foschini–Miljanic algorithm with time-dependence

Consider the system

\[
\frac{dp_i(t)}{dt} = k_i \left( -p_i(t) + \frac{\gamma_i}{G_{ii}(t)} \left( \sum_{j \neq i} G_{ij}(t) p_j^d(t) + \nu_i \right) \right)
\]

with the link gains evolving according to

\[
G_{ij}(t) = G_{ij}(0) (1 + S_{ij}(t)),
\]

where the \( S_{ij} \) are linear interpolants of the discretized independent Wiener processes with saturation

\[
\tilde{\mathcal{W}}_{ij}(t_{k+1}) = \min \left( \max \left( \tilde{\mathcal{W}}_{ij}(t_k) + \tau N_{ij}(k), -0.49 \right), 0.49 \right).
\]
We can then use the bounding nonlinearities

\[
J_i^{\text{min}}(p) = \frac{\gamma i}{G_{ii}(0)} \left( \sum_{j \neq i} \frac{1}{3} G_{ij}(0) p_j + \frac{2}{3} \nu_i \right)
\]

and

\[
J_i^{\text{max}}(p) = \frac{\gamma i}{G_{ii}(0)} \left( \sum_{j \neq i} 3 G_{ij}(0) p_j + 2 \nu_i \right),
\]

provided that we consider parameter values that allow these to have positive equilibria.
(a) $p_2$ against $p_1$; other initial data 0.4.

(b) $p_3$ against $p_1$; other initial data 0.4.

(c) $p_4$ against $p_1$; other initial data 0.4.

(d) $p_2$ against $p_1$; other initial data 2.

(e) $p_3$ against $p_1$; other initial data 2.

(f) $p_4$ against $p_1$; other initial data 2.
Figure: Time evolution of the value of the component $p_1$ for trajectories through 100 different data values.
Figure: Time evolution of the four components of the signal-to-interference ratio.
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Under relevant monotonicity and scalability conditions, if there is any trajectory for which an appropriate boundedness result holds, then this trajectory is uniformly asymptotically stable.

Further, then every trajectory tends uniformly to this one.

So in fact then every trajectory is bounded, and the asymptotic behaviour can be predicted by simply studying that of a convenient class of trajectories.

Our auxiliary result provides a quick way to check whether there is a trajectory satisfying the appropriate boundedness result, for certain systems.
Thank you for your attention!

Slides can be found on http://www.damtp.cam.ac.uk/user/esmd2/talks.html.