Hollywood likes science. Or, to put that more precisely, Hollywood likes to sound as if it likes science. Consequently, what Hollywood really likes are scientific buzzwords - words that sound technical and important so that they impress the audience. In the realm of buzzwords, it is hard to think of any more prominent or more fitting of the description than the words “quantum” and “chaos”. Behind each of these simple words lies a vast mass of twentieth-century mathematics. Between them, they arguably represent two of the greatest achievements on the applied side of subject of recent times. In the space below, we look to reveal just some of this remarkable science and history from behind the buzzword veil. Remarkably, though, we can go even further than this. In recent decades mathematicians have begun to investigate the possibility of combining the two - of elucidating the precise ways in which the theory of chaos manifests itself in the quantum regime. The result is the very active modern research topic of quantum chaos, an amalgamation of two such potent scientific buzzwords that is sure to please the mathematician and the Hollywood producer alike!
The uncertainty principle will not help you now, Stephen. All the quantum fluctuations in the universe will not change the cards in your hand. I call. You are bluffing. And you will lose!


Well, actually, nobody on this planet ever really chooses each other. I mean, it’s all a question of quantum physics, molecular attraction, and timing.

Annie Savoy (Susan Sarandon), Bull Durham (1988).

Then I saw little Tiffany. I’m thinking, y’know, eight-year-old white girl, middle of the ghetto, bunch of monsters, this time of night with quantum physics books? She about to start some shit, Zed. She’s about eight years old, those books are way too advanced for her. If you ask me, I’d say she’s up to something.


The signal pattern is learning, it’s evolving on its own, and you need to move past Fourier transforms and start thinking quantum mechanics . . .

Maggie Marsden (Rachael Taylor), Transformers (2007).

You’ve never heard of Chaos theory? Non-linear equations? Strange attractors? Dr. Sattler, I refuse to believe you’re not familiar with the concept of attraction.

Dr. Ian Malcolm (Jeff Goldblum), Jurassic Park (1993).

You’ve got to listen to me. Elementary chaos theory tells us that all robots will eventually turn against their masters and run amok in an orgy of blood and kicking and the biting with the metal teeth and the hurting and shoving

Professor Frink (Hank Azaria), The Simpsons (1994).

I’m sorry about the mess. Sometimes I think my bunkmate majored in chaos theory.

Temporal Agent Daniels (Matt Winston), Star Trek: Enterprise (2001).

Scientists tell us that the world of nature is so small and interdependent that a butterfly flapping its wings in the Amazon rainforest can generate a violent storm on the other side of the earth. This principle is known as the Butterfly Effect. Today, we realize, perhaps more than ever, that the world of human activity also has its own Butterfly Effect - for better or for worse.

1 Quantum Mechanics

The theory of quantum mechanics, to which the buzzword quantum refers, really began with Albert Einstein’s Nobel Prize-winning paper of 1905, *On a Heuristic Viewpoint Concerning the Production and Transformation of Light*. This paper provided the first explanation of a phenomenon known as the photoelectric effect, in which the shining of a light onto certain metallic surfaces is observed to stimulate emission of electrons from that surface. In particular, the number of electrons emitted (as measured by the current measured in a test circuit) is observed to depend not on the intensity of the light, as many theories of the day suggested, but instead on its frequency. Einstein’s revolutionary idea was that the light, which had always been viewed as a wave, could also exhibit some particle characteristics. So, in effect, the photoelectric effect could be resolved by viewing the light as a stream of particles, called photons.

The reason this paper had such a profound effect lies in its paradigm-shifting idea: that waves can, in some circumstances, behave as particles. This inspired the French physicist Louis de Broglie to introduce the fundamental tenet of modern quantum theory, that particles can behave as waves too! Niels Bohr, Werner Heisenberg, Erwin Schrödinger and many others ran with this idea, and so our theory was born.

To understand Schrödinger’s key idea, we require Max Planck’s formula for the energy of a photon: \( E = h\omega \) and de Broglie’s formula (derived from this through arguments of special relativity) for its momentum: \( |p| = \hbar|\mathbf{k}| \). Schrödinger’s idea was then to consider the plane wave form for a ray of light: \( \psi(t, x) = \exp\left[-i(\omega t - \mathbf{k} \cdot \mathbf{x})\right] \). Now, direct differentiation and application of the Planck and de Broglie formulae give us the equations

\[
\frac{i\hbar}{\partial t} \psi = \hbar\omega \psi = E\psi
\]

and

\[
\frac{\hbar}{i} \nabla \psi = h\mathbf{k} \psi = \mathbf{p} \psi.
\]

Now Schrödinger’s revolutionary idea was that we should regard these equations as fundamental. The consequence of this is that we now view these equations as governing the quantum mechanics of all particles, rather than just photons. In particular, by completing the system using the classical relation for the energy of a particle of mass \( m \) in a potential \( V(x) \), \( E = |p|^2/2m + V(x) \), we obtain the Schrödinger equation

\[
\begin{align*}
\frac{i\hbar}{\partial t} \psi &= -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi.
\end{align*}
\]

This is the key equation in all of quantum mechanics, as it, with the appropriate boundary conditions, completely describes the quantum wave function of an individual particle of mass \( m \) moving in a general potential \( V(x) \). It is such systems that are commonly studied in chaos theory, and so are of interest to us here.

This equation can be separated as \( \psi(t, x) = T(t) \Psi(x) \), giving the simple time-evolution equation

\[
\frac{i\hbar}{\partial t} T = ET
\]

and, what is most relevant here, the time-independent Schrödinger equation

\[
-\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi = E \Psi,
\]

or, in an alternative form

\[
H \Psi = E \Psi
\]

where \( H \) is the Hamiltonian operator of the system. This is an eigenvalue equation, with the eigenvalues giving the allowed energy levels. It is the crucial equation for studies of quantum chaos.

The theory of quantum mechanics now consists of analysis of this equation as applied to various systems. For our purposes,
however, this is as far into this immense subject as it is necessary to go. Suffice to say, however, that the agreement between experiment and the many remarkable conclusions that can be drawn from this theory is truly exceptional and places the theory of quantum mechanics as one of the most successful scientific theories of all time.

Aside: The Heisenberg Uncertainty Principle

Interviewer: How does the Heisenberg Compensator work?
Michael Okuda (Star Trek): It works very well, thank you very much.
Time Magazine (1994).

Heisenberg’s Uncertainty Principle seems to crop up in films and television programmes almost as frequently as quantum mechanics itself. To understand the Uncertainty Principle, we must first consider the true mathematical structure of quantum theory.

Along with wave functions, the other important quantities in quantum theory are known as operators. Operators are functions that act on the Hilbert space of all relevant wave functions. The final thing that is needed is an inner product, for which we use the canonical inner product

$$\langle \psi | \phi \rangle = \int \bar{\psi} \phi \, dx.$$ 

Now, with respect to this inner product, we may define observables, which represent a physical quantity that can be measured, as self-adjoint linear operators. We also restrict our attention to wave functions that are normalizable, in that their modulus with respect to this inner product is finite. These observables and normalizable wave functions form the mathematical formulation of quantum mechanics.

In this formulation, with everything being statistical in nature, all observables, $A$, have two important physical attributes with respect to the wave function that represents the state that the system is in, analogous to the mean and the standard deviation in statistical theories. These are the expectation

$$E_\psi(A) = \langle \psi | A \psi \rangle$$

and the dispersion

$$\Delta_\psi(A) = \left( E_\psi(A^2) - E_\psi(A)^2 \right)^{\frac{1}{2}}.$$

For consideration of the Uncertainty Principle, the two relevant observables are the position operator and the momentum operator, defined in one dimension by $X_\psi = x\psi$ and $P_\psi = \frac{i}{\hbar} \frac{\partial \psi}{\partial x}$ respectively. However, we do not use exactly these observables, but rather form $A = X - E_\psi(X)$ and $B = P - E_\psi(P)$, and we consider a variable $s$ in $\mathbb{R}$. Both $X$ and $P$ are self-adjoint, meaning that $(A - isB)^* = A + isB$. This allows us to rewrite the positivity of the norm inequality, $\langle (A - isB)\psi | (A - isB)\psi \rangle \geq 0$, as

$$\langle \psi | (A + isB)(A - isB)\psi \rangle \geq 0,$$

or in expanded form (noting that the operators do not commute)

$$E_\psi(B^2)s^2 - \hbar s + E_\psi(A^2) \geq 0.$$

Therefore the discriminant of this quadratic must be at most zero, giving us the inequality

$$E_\psi(A^2)E_\psi(B^2) \leq \frac{\hbar^2}{4}.$$ 

Now, we note that the expectations here are precisely alternative ways of writing dispersions squared, so that this result can be square-rooted to give

$$\Delta_\psi(X)\Delta_\psi(P) \leq \frac{\hbar}{2}.$$ 


This is the Heisenberg Uncertainty Principle, which says, pretty much exactly as they tend to say in movies, that the more accurately one measures the position of a particle, the less accurately one can simultaneously know its momentum, and vice versa. This produces a fascinating limitation on our knowledge, which is of great interest to scientists and movie producers alike.

2 Chaos Theory

So, now for chaos. The theory of chaos really goes back to Newton’s attempts on the Kepler Problem of describing planetary motion. Though one of his greatest achievements was the full solution of the two-body problem, he soon realized that the three-body (e.g. Sun, Earth and Moon) version was much more difficult. Indeed, much effort over the following decades showed it to be virtually impossible to obtain formulae to describe the exact motion of the three bodies. Such a counter-intuitive problem in an entirely deterministic system lies at the heart of chaos. It was Henri Poincaré who got us started on the path to the modern theory by suggesting that, rather than try to find exact formulae, we should instead look at the qualitative aspects of such systems. He suggested that we should be asking questions like “What happens to body A as \( t \to \infty \)?”. Poincaré’s other key contribution to the field was his clarification that dynamical systems should be studied not just by looking at the particle position, but by plotting a phase portrait consisting of both the position and the momentum. Any path of points in the phase portrait corresponding to a particular set of initial conditions is known as a trajectory. In all but the simplest cases, the dimension of the phase space in which this resides makes it difficult to visualize. For this reason, investigators tend to utilize a Poincaré section, which means taking a slice through phase space (essentially strobing the system to see what’s happening every time a particular condition, which we specify, is satisfied) to reduce its dimension by two.

Since the investigation of chaotic systems requires the analysis of the long-term evolution of a multitude of trajectories, it is no surprise that most of the major developments in the history of chaos theory have happened since the dawn of the computer age. Among these, the major player was the American mathematician Ed Lorenz, who can be considered as the founder of the modern theory. Lorenz’ work helped formulate the accepted modern definition of chaos: aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

This deserves some clarification. By aperiodic long-term behavior we mean as \( t \to \infty \) there are some trajectories that neither move off towards infinity nor settle down to any fixed point or periodic trajectory. In addition, sensitive dependence on initial conditions means that two trajectories that start with similar initial conditions (e.g in similar places heading the same direction) will diverge at an exponential rate. A key word in chaos theory is eventually - a lot of the initial behavior is often neglected in favor of the asymptotic long-time patterns. We see now why Poincaré’s question was so important . . . .

Using this definition, mathematicians are able to classify all dynamical systems into
Figure 1: The behavior of similar shots on the first two billiards discussed in the text. On the square billiard the similar initial conditions give rise to trajectories that remain the same distance apart at all times, whereas once the inner circular rail is added, the two initially similar trajectories rapidly diverge.

one of three natural classes of dynamical system, which are called chaotic, integrable and mixed. Chaotic systems are ones in which the entire phase space exhibits chaotic behavior, while, at the other end of the scale, integrable systems are those that display no chaos whatsoever. The systems in between these two extremes, where some but not all of the phase space is chaotic, are called mixed systems. These distinctions turn out to be important in the realm not only of classical chaos but of quantum chaos as well.

Dynamical systems that exhibit chaos can, and usually do, turn out to be horrendously complicated. Therefore, for studying the foundations of the subject and laying the groundwork for an investigation of quantum phenomena, it makes sense to simplify what we want to consider as much as possible. It turns out that arguably the simplest (and certainly the easiest to understand from an intuitive point of view) such family of systems is the family of two-dimensional billiard systems. By a billiard system, we mean an idealized billiards / pool / snooker (delete as you think appropriate) table without any pockets and without any friction, together with a single ball. This ball is then struck as the cue ball in any of the aforementioned sports, and its motion around the table over time is observed. What we end up with is a single particle moving freely in a region whose shape is pre-defined, and can be changed to alter the dynamics. When the particle reaches any point on the boundary, it bounces back in such a way that the angle of reflection equals the angle of incidence, while the speed of the particle remains constant. Terminologically, we shall still refer to the particle as the ball, to the boundary as the rails, and to the domain on which the motion occurs as the table.

To simplify matters still further, we can simply fix a particular initial strike speed so that all shots on the table are hit with the same speed. This way, the particle has the same magnitude of momentum in all trajectories, so we can neglect the momentum and consider just the particle position to investigate the billiard dynamics.

First of all, consider the simple case of a square billiard. Now we imagine any shot
on the table and the consequent trajectory that the particle follows around the table. If the initial point is displaced and the shot now taken from this new initial point at the same angle as before, we see that the particle follows a new trajectory that remains a constant distance from the previous one, as shown in Figure 1. Alternatively, if the initial conditions are changed not by displacing the initial point, but by changing the angle of the shot, we see that the new trajectory diverges from the original one at a linear rate. The conclusion we draw is thus that the divergence of what we call “nearby” trajectories from one another is at most linear, and therefore this billiard system is entirely integrable.

Now let us add a circular rail to the center of the previous billiard. By this simple change, we dramatically alter the relationship between pairs of nearby trajectories. To see this we consider the simplest possible trajectory, that bouncing back and forth perpendicularly between one edge of the outer square rail and the inner circular one. If we now change either the initial position or the initial direction of the particle, then clearly the motion on first approach toward the circular rail is no longer perpendicular to it. Therefore, the reflection off the inner rail changes the angle of the particle, and so the new trajectory is completely different. Indeed, numerical simulations, such as that on which the right-hand diagram in Figure 1 is based, show that small perturbations of the initial conditions of any trajectory will result in a completely different one; this is sensitive dependence on the initial conditions. Therefore, this system is now completely chaotic.

The most generic, and indeed interesting, type of system though is that which exhibits characteristics of both of the previous two, namely the mixed system. To obtain a billiard that gives us a mixed system, we need to make much more substantial changes to the geometry. The most intuitive example is the so-called mushroom billiard, whose geometry consists of a rectangular stem attached to semicircular cap. On this table, it is not hard to see the presence of both integrable and chaotic paths. For the first, think simply of the so-called bouncing ball modes (as long as they are far enough down the stem), which move perpendicular to the “vertical” sides of the stem, or for a less trivial version, those paths that bounce around in the stem before moving into the cap but colliding with the top of the mushroom close enough to the center-line that they are reflected back into the stem without entering the rest of the cap (they must also avoid hitting the “vertical” stem rails too close to the cap). Now, to see that chaotic trajectories are also possible, imagine starting in the far left of the cap at such an angle that the first impact with any rail is with the straight rail at the bottom of the cap, to the left of the stem. Most such trajectories will oscillate around within the cap for a while before eventually entering the stem, then returning to the cap, . . . . Now suppose that the initial angle is changed very slightly. Initially, while within the cap, the convex nature of the upper rail keeps the divergence from the previous trajectory at most linear. However, it is simple to show by means of numerical simulations that the point at which the new trajectory leaves the cap can now become vastly different, whereafter the two trajectories will bear no similarity to each other whatsoever and will diverge rapidly and greatly - symptomatic of the sensitive dependence characteristic of chaos.

So there we are, not only can chaos researchers hear mention of their fascinating subject in mainstream popular media, but they can also now do their research in their local snooker hall!
Aside: The Butterfly Effect

A butterfly can flutter its wings over a flower in China and cause a hurricane in the Caribbean. They can even calculate the odds.
Jack Weil (Robert Redford), Havana (1994)

The Butterfly Effect is the popular embodiment of chaos theory. So much so, in fact, that Hollywood writers Eric Bress and J. Mackye Gruber even made a film named after it. The explanation offered in the quote above explains adequately what is meant by the butterfly effect, but a question of interest is why such an effect should exist at all.

Hollywood tends to use the Butterfly Effect as a trope for conveying the vagaries of time travel, explaining how just the subtlest change to the past can have major implications on the future of the Universe. In this idea, we recognize the presence of sensitive dependence on initial conditions; if a very slight change were made to one person’s life in the Roman times, for instance, our world today could well be very different. Extreme examples of this would be changing events such that two people never met, one man avoided boarding a doomed ship, etc.. This chaotic behavior of the time-evolution of society would appear to be one of the main problems with any “normal” theory of time travel. For such a theory to be self-consistent, it seems that one would have to invoke the use of concepts such as parallel universes to avoid inconsistencies and paradoxes.

Anyway, the true Butterfly Effect has nothing whatsoever to do with time travel. It is in fact a meteorological phenomenon which has to do with the unpredictability of the behavior of our atmosphere. Like chaos theory itself, the history of the Butterfly Effect really begins with Poincaré, who predicted, once he had formulated the first real understanding of chaos, that such phenomena may be common in meteorological systems. However, due to the lack of computing power available, this idea was not really taken any further until 1961, when Lorenz was investigating computational simulations of the weather. In one of these, he reran the simulation with one of the initial values set at 0.506 rather than the 0.506127 that he had used in the initial run. The fascinating result was that the weather system generated was now completely different from the original. These findings, published by Lorenz in 1963, upheld Poincaré’s claim and showed the presence of chaos in our weather system. Interestingly, Lorenz initially used a seagull rather than a butterfly as his illustrative creature, though that was soon changed, and by the time of the 1972 meeting of the American Association for the Advancement of Science, Lorenz had arrived at the immortal lecture title “Does the flapping of a butterfly’s wings in Brazil set off a tornado in Texas?”.

Redford’s line above does, though, contain one major flaw in its implications. By the ending “They can even calculate the odds”, the suggestion appears to be that this is a predictable phenomenon. This could not be further from the truth! The consequences of the Butterfly Effect, or more precisely of the chaotic behavior of our weather system, are important and extreme. The weather depends upon many factors: temperature, humidity, land profile, . . . , and we now know that each one of these must be known exactly in order for the future evolution of the system to be predictable. Obviously, even with modern measurement techniques, this is impossible. Therefore, we cannot predict the weather very well, and it seems very
unlikely that we will ever be able to do so.

The way the weather system works, like many chaotic ones, is that there is an initial transient phase, in which similar sets of initial conditions evolve in similar ways. This corresponds to our short-range forecasts; our relatively accurate initial measurements allow us to predict the weather for a few days with reasonably good accuracy. However, the evolutions typically then tend to diverge rapidly and are very quickly very different from one another. This is where weather prediction becomes extremely difficult; the inaccuracies in our initial measurements have been amplified to the extent that what we can predict from them may now be completely different from what actually happens. In fact, because of this, meteorologists often use other factors, such as long-term statistical comparisons, to complement their longer-range predictions to hopefully improve their reliability. Yet, inevitably, they still get some drastically wrong, and when they do, they tend not to hear the end of it!

3 Quantum Chaos

We have now seen that there is a predominance of chaos in macroscopic classical dynamical systems, and that the laws of quantum mechanics govern the dynamics of systems on the atomic scales. Consequently, it makes sense for us to ask what happens in the event of a merger of these two - what happens when we take a classically chaotic system and view it from a quantum perspective? Once again, we have arrived at a question first considered by Einstein, in 1917, in one of his less heralded papers. However, it wasn’t really until the work of such modern luminaries as Berry and Bunimovich in the 1970s and 1980s that this field really got off the ground. Since then, however, it has been (and remains) a fervent ground for mathematical and physical research.

Intuitively, it seems clear that there must be some quantum manifestation of chaos, as we can imagine (as we indeed have been doing) the various billiard systems as two dimensional analogues of the usual particle in a box problem, for which there are well known quantum solutions. The problem comes when we begin to investigate what exactly these manifestations are. We imagine a general quantum system of the type discussed in Section 1, described by its Schrödinger equation and appropriate boundary conditions. The most natural guess would be that chaos would appear in the quantum realm in much the same way as it appears in the classical one, namely in the form of sensitive dependence on initial conditions. Alas, the truth is rarely pure and never simple, so the saying goes, and nowhere is this more appropriate than in quantum mechanics. The reasons for the failure of this guess arise from equation 2 from above. Its result tells us that the time-evolution of our system from the given initial state $\psi_0$ is governed by the relation

$$\psi = e^{-\frac{iHt}{\hbar}} \psi_0$$

where $H$ is the Hamiltonian of the system. The pertinent point here is that the exponential operator is unitary and depends only on the Hamiltonian and the time. This severely limits the possible divergence of two states that are initially very similar, prohibiting any possibility of anything that we might regard as satisfying the definition of sensi-
tive dependence on initial conditions. Consequently, the manifestations of chaos in the quantum regime must appear in more subtle, less overt, ways.

So it turns out. To see (some of) the true connections between chaos and quantum mechanics, we must first go back to our classical description of the system to be investigated. In particular, we consider so-called Hamiltonian systems, those whose phase space is entirely described by a quantity called a Hamiltonian and a set of equations known as Hamilton’s equations, of which billiards comprise a subset. The approach we then take, in general, is to quantize this Hamiltonian, essentially by replacing classical positions and momenta with the relevant quantum position and momentum operators (in one-dimension, these were seen above to be \( X = x \) and \( P = \frac{\hbar}{i} \frac{\partial}{\partial x} \)). We can then substitute the resulting operator into equation 3 to obtain an eigenvalue equation whose eigenvalues will give us the energy levels of the system. As so often in quantum theory, it is the energy levels of the system that we care most about here. More specifically, according to the groundbreaking work of Wigner, Berry and Robnik, it is the statistical distribution of these levels that is of interest.

Obviously, to generate any sort of meaningful statistical distribution requires the accurate calculation of large numbers of these eigenenergies. As recently as 2008, Alex Barnett and Timo Betcke completed the first study of a mixed system that achieved this goal. The system in question was the quantum version of our mushroom billiard system. To quantize this, the simplest approach is to use intuition. We know that the particle moves freely within the domain, so there the potential must be 0, while at the boundaries of the domain (the rails) it is reflected, so that the potential must be infinite. Using the standard Hamiltonian of a free particle (so \( V = 0 \)) and desymmetrizing for technical reasons, we formulate this quantum mechanically as follows:

\[
-\frac{\hbar^2}{2m} \nabla^2 \psi = E \psi
\]

subject to \( \psi = 0 \) on the boundary of the half-mushroom (desymmetrized by halving it along its vertical axis of symmetry). We then regard this as a time-independent Schrödinger problem and seek primarily the eigenvalues to give the energy levels of the quantum mushroom billiard.

The novel approach used by Barnett and Betcke was to generate a basis set of particular solutions, \( \{ \phi_i \} \), that individually satisfy the equation for some trial energy value but not the boundary condition. They then used variables \( x_i \) to form the linear combination \( \sum_i x_i \phi_i \) and essentially minimized the maximum size of this across the boundary over all possible \( x_i \) for numerous trial values of the energy. Extrapolating a relationship, they were able to obtain good, reliable approximations to the eigenvalues by finding those values of \( E \) for which the aforementioned minimum was below a certain threshold quantity. There are many intricacies to the precise numerics involved, including how the appropriate basis functions are to be chosen; for further information on these, their paper (given in the bibliography below) provides excellent detail.

This method gives the energy levels for the odd eigenmodes - those that are not symmetric about the original mushroom’s axis of symmetry. The level spacings can then be obtained simply by taking the differences between the levels. These are then unfolded to unit mean energy by dividing each by the mean of all the energy levels determined. In this way, Barnett and Betcke were able to obtain the so-called nearest-neighbor spacing distribution (NNSD) for the first 16601 levels, much further than anyone had previously gone for a mixed system. Before presenting their results, we must first summarize the most important pre-existing knowledge in this field, namely the theoretical pre-
Figure 2: Histograms of the NNSDs for the first 16601 modes of the quantum mushroom billiard. Theoretical predictions, together with ±1 standard error, are also shown. (a) shows all modes against the Berry-Robnik prediction. (b) shows just the regular (integrable) modes against the Poissonian prediction. (c) shows all ergodic (in this case the same as all chaotic) modes against the Wigner prediction. (d) is like (c) but with some trivial bouncing ball modes removed.

dictions of these NNSDs. Crucial in these breakthroughs was the elucidation, through the work of mathematicians such as Hermann Weyl and Eugene Wigner, of an amazing link between the subject and random matrix theory. This allowed researchers to apply analytical results from the theory of random matrices to problems of quantum chaos. In this way, investigators were able to show that the NNSDs of quantizations of classically integrable systems should exhibit ordinary Poissonian statistics, while those of classically chaotic systems should follow a new distribution, now known as the Wigner distribution

\[ P(S) = \frac{1}{2\pi} Se^{-\frac{S^2}{4}}. \]

Following on from this, and based on their meticulous analytical work, Sir Michael Berry and Marko Robnik conjectured that the NNSDs of mixed systems should follow a distribution that interpolates between the Poissonian and Wigner distributions, in a
particular way, based on the proportion of phase space that is chaotic. This remains an open problem, known as the Berry-Robnik conjecture. One of the main advantages of the quantum mushroom billiard system is that its phase space is cleanly separated, so that the fraction of chaotic phase space can easily be calculated. It also allows for the possibility of using one system to check the validity of all three of these theoretical statistical distributions. This is precisely what Barnett and Betcke did; their results are shown in figure 2.

Therefore, due to this and other data collected, it would appear that one way in which chaos manifests itself in the quantum regime is in the precise nature of the system’s energy level spacing distribution. Essentially the approach outlined above consists of studying the behavior of the system at extremely high energies. For these systems, taking this high-energy limit can be shown to amount to the same thing as taking the effective Planck’s constant for the system to be tending towards zero. By definition, then, this is a semiclassical approach to the problem. It should be noted that this is just one of several different ways of approaching quantum chaos that are in active use, though at the time of writing this semiclassical approach would seem to be the most promising.

Epilogue

So, what exactly does our Hollywood producer get from all these developments in quantum chaos, then? Well, at the moment, one would have to say nothing; the work so far is primarily of interest only to our mathematician. But as with any relatively young subject, this is a rapidly growing field, and mathematicians over the next few decades have the potential to make great strides within it. On the horizon are two tantalizing possibilities, both of which might well be of interest to both our producer and our mathematician.

The first of these pertains to the potential experimental applications of the theory. For several years now, scientists have been investigating the feasibility of the quantum computer. At the present, we remain some distance from a full-scale working prototype. However, one of the most promising candidates for use in solid-state quantum computation is the quantum dot, a tiny semiconducting crystal whose conducting properties depend upon its size and shape. The relevance of quantum dots here is that they behave essentially as practical models of quantum billiard systems. Consequently, a thorough understanding of the behavior of quantum billiards, both chaotic and non-chaotic, is crucial in our quest to further understand and develop the quantum dot. This is a quest that, if successful, may one day help to revolutionize computers and greatly advance our computational ability.

The second of the tantalizing possibilities relates to the outstanding problem in mathematics today: the Riemann hypothesis. Though the Riemann hypothesis resides in the somewhat different field of analytic number theory, an intriguing link has been noticed between it and the field of quantum chaos. In the early twentieth century, Hilbert and Polya conjectured that the zeros of the Riemann zeta function correspond to the eigenvalues of some self-adjoint linear operator. Over 70 years later, Berry turned his
attention to the question of whether these two eminent figures in mathematical history could be correct. Upon studying the data, he concluded that “exactly this behavior would be expected if the imaginary parts of the zeros were eigenvalues . . . of the Hamiltonian operator obtained by quantizing some still unknown dynamical system whose phase-space trajectories are chaotic”. If one were able to find such a system, then the field of quantum chaos might just be able to provide the key to resolving the greatest problem of our time.

References


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