Binary Stars

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[For Examples class 1-3pm Monday 11th February] Lent Term, 2019

Examples Sheet I

1. A binary star system with a period of 4 yr lies at 200 pc from the Sun. The projected orbit of each component holding the other fixed is an ellipse. For each star the semi major axis of the ellipse passes through the companion and the apastron separation is four times the periastron separation. The semi-major axis subtends 20 mas and the semi-minor axis 8 mas. Find the eccentricity, inclination, semi-major axis and total mass of the system.

2. Two stars of equal surface temperature form an eclipsing binary system. The orbit is circular. If the stars are spherical and limb-darkening can be ignored, show that both eclipses are the same depth in magnitudes and that this depth cannot exceed a certain amount. Find this amount.

3. In a single-lined spectroscopic binary in an eccentric orbit the eccentricity e, the longitude of periastron ω_{p}, the orbital period P and the semi-amplitude K_1 of the radial velocity of star 1 can be measured. When the unknown inclination of the orbit is i, show that the mass function defined as

$$F_1(M_1, M_2, i) = \frac{M_2^3 \sin^3 i}{(M_1 + M_2)^2} = \frac{P(1-e^2)^{3/2}}{2\pi G} K_1^3,$$

where M_1 and M_2 are the masses of star 1 and star 2 respectively. Show that the observations give a lower limit to M_2. How might we set limits on i?

4. A total of N stars are placed at random on the celestial sphere. Show that the probability distribution for the angular separation of nearest neighbours is

$$P(\theta)\, d\theta = \frac{N-1}{2^{N-1}} (\sin\theta)(1+\cos\theta)^{N-2}\, d\theta.$$

The restricted bright star catalogue contains 4908 objects brighter than $V = 6$. Of these 114 are doubly bright visual systems with separations less than $3.5\,\mu$rad. What would be the most likely angular separation of any star's nearest neighbour if all the stars were single and placed at random? What is the probability that all 114 double stars are simply random superpositions?

5. The Salpeter distribution for stellar masses for the fraction of stars with masses between M and $M + dM$ takes the form

$$\phi(M) = \begin{cases} kM^{-\alpha}, & M > M_0, \\ 0, & M < M_0, \end{cases}$$

with $\alpha \approx 2.35$.

If the two stars in all binary systems are chosen at random from this distribution show that, for a mass ratio $q > 1$, the number of system with mass ration between q and $q + dq$, $n(q)\,dq \propto q^{-\alpha}\,dq$. What is the distribution of q for $q < 1$?

6. Suppose Newton's law for the force between two point masses of total mass M and separation $\mathbf{r}$ is perturbed to

$$\ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r} + \mathbf{f}.$$

Show that the energy E, specific angular momentum $\mathbf{h}$ and Laplace-Runge-Lenz vector $\mathbf{e}$ change according to

$$\dot{E} = \mu\dot{\mathbf{r}}.\mathbf{f}, \quad \dot{\mathbf{h}} = \mathbf{r} \times \mathbf{f} \quad \text{and} \quad GM\dot{\mathbf{e}} = \mathbf{f} \times \mathbf{h} + \dot{\mathbf{r}} \times (\mathbf{r} \times \mathbf{f}),$$

where $\mu = M_1 M_2 / M$ is the reduced mass and M_1 and M_2 the individual masses of the stars.

7. Show that in a frame in which all the material is corotating with angular velocity Ω the equation of hydrostatic equilibrium of a star can be written as

$$\nabla P = -\rho \nabla \phi$$

for pressure P, density ρ and combined gravitational and centrifugal potential $\phi(\mathbf{r})$ which satisfies

$$\nabla^2 \phi = 4\pi G\rho - 2\Omega^2.$$

Show that P and ρ must be constant on equipotential surfaces. Hence deduce that $\nabla^2\phi$, but not necessarily $|\nabla\phi|$, constant on equipotential surfaces.

Argue that, for a star of uniform composition, temperature T is also constant on equipotential surfaces.

The star is in radiative equilibrium with heat flux

$$\mathbf{F} = -\chi \nabla T = -\chi \frac{dT}{d\phi} \nabla \phi,$$

where χ is the conductivity which is related to the opacity $\kappa(\rho, T)$ by

$$\chi = \frac{4acT^3}{3\kappa\rho},$$

where a is the radiation constant and c is the speed of light. Show that the effective temperature on the surface of the star

$$T_{\mathrm{e}} \propto g^{1/4},$$

where g is the magnitude of the effective gravitational acceleration. Sketch the cross-section of a rapidly rotating star and indicate where it is hottest.

Why is it not in general possible for the energy balance to be given simply by

$$\nabla.\mathbf{F} = \rho\epsilon,$$

where $\epsilon(\rho, T)$ is the energy generation rate per unit mass?

Now suppose that there is a steady circulation velocity field $\mathbf{v}(\mathbf{r})$ so that the energy balance is given instead by

$$\rho T \frac{ds}{dt} = \rho \mathbf{v}.T\nabla s = \rho\epsilon - \nabla.\mathbf{F},$$

where $s(\rho, T)$ is the specific entropy. Use continuity and the thermodynamic relation

$$T\, ds = dh - \frac{1}{\rho}\, dP,$$

where $h(\rho, T)$ is the specific enthalpy, to show that

$$\int_S \mathbf{F}.\mathbf{dS} = \int_V \rho\epsilon\, dV,$$

where S is an equipotential surface enclosing volume V.

Hence show that the radiative gradient is given by

$$\frac{d\log T}{d\log P} = \frac{3\kappa PL}{16\pi acGmT^4}\left(1 - \frac{\Omega^2 V}{2\pi Gm}\right)^{-1},$$

where L is the rate of energy generation within V and m is the mass in V.

Estimate the factor $\frac{\Omega^2 V}{2\pi Gm}$ at (a) the surface and (b) the centre of one component of an equal-mass binary system consisting of two stars like the Sun whose central density $\rho_{\mathrm{c}} \approx \bar{\rho}$, where $\bar{\rho}$ is the mean density, when the components are just filling their Roche lobes and comment.