Binary Stars

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Examples Sheet II

1. A white dwarf behaves as an $n = 3/2$ polytrope with radius $R = \gamma M^{-1/3}$ and isotropic moment of inertia $I = Mk^2R^2$, where M is its mass and k and γ are constants. It accretes from a main-sequence companion via an accretion disc so that material is accreted with the specific angular momentum of a Keplerian orbit at the surface of the star. Show that the total angular momentum accreted along with a mass ΔM is

$$\Delta J = C\left\{(M_0 + \Delta M)^{4/3} - M_0^{4/3}\right\},$$

where C is a constant to be determined.

Suppose that before it accretes the spin of the white dwarf is negligible and that it maintains a uniform angular velocity $\Omega(t)$ while it accretes. From models of white dwarfs $\gamma = 0.0126\, R_\odot M_\odot^{1/3}$ and $k^2 = 0.205$. How much material can a white dwarf of $0.5\, M_\odot$ accrete before it is spun up to its break up velocity?

2. Main-sequence stars with mass $M > 1.5\, M_\odot$ have a convective core of mass $M_\mathrm{c} \approx \frac{1}{4}M$ at zero age when the uniform hydrogen abundance by mass is $X = 0.7$. The size of the core falls linearly with X as the star evolves until, at the end of the main sequence when $X \approx 0$, $M_\mathrm{c} \approx \frac{1}{10}M$. Assuming that such stars burn hydrogen at a constant luminosity $L \propto M^3$ find M_c/M as a function of t/τ_MS, where t is the age and τ_MS is the total main-sequence lifetime.

At $t = \frac{9}{21}\tau_\mathrm{MS}$ a star of $2\, M_\odot$ accretes $3\, M_\odot$ of unprocessed material from a binary companion in a time that is short compared with the nuclear burning timescale. The material accretes on to the surface of the star without mixing into its core. Once the star has regained equilibrium find its effective age in the form $t = \alpha\tau_\mathrm{MS}$, that is as a fraction of the main-sequence lifetime of what is now a $5\, M_\odot$ star.

Its evolution then continues unaffected by the companion until it exhausts its central hydrogen. What would be the mass of an isolated star that would have evolved to the same point in its evolution at this time?

Comment on this in relation to blue stragglers in middle-aged star clusters.

3. Two stars of masses M_1 and M_2 move in circular orbits about their centre of mass. What is (i) the orbital angular momentum J_1 of star 1 and (ii) the total orbital angular momentum J, in terms of the masses and the orbital period P?

Wind from star 1 reduces M_1 at a rate which is steady and slow compared with P. The wind leaves the stellar surface in a spherically symmetric manner and can be assumed not to interact with star 2. Justify briefly the equation

$$\dot{J} = \frac{J_1}{M_1}\dot{M}_1$$

for the evolution of total orbital angular momentum and show that this implies

$$P \propto (M_1 + M_2)^{-2}\,.$$

Now suppose that (a) a constant fraction f of the mass lost from star 1 is accreted by star 2, (b) the remaining $1 - f$ escapes to infinity with the same specific angular momentum as before and (c) the intrinsic spin of both components remains negligible compared to the orbital angular momentum. Show that the variation of P is now given by

$$P \propto M_1^{-3f} M_2^{-3} (M_1 + M_2)^{-2}\,.$$

Discuss qualitatively the validity of each of assumptions (a) to (c) and in particular their possible dependence on orbital separation (relative to star size) and on wind velocity (relative to orbital velocity).

Finally, suppose that the period is so long, perhaps $10^6 - 10^7$ yr, that the mass loss by stellar wind takes place in a time which is *short* compared with the orbital period. Without detailed calculation, what would you expect the effect on the orbit to be?

4. In a semi-detached binary, with conservative Roche lobe overflow (RLOF), show that the radius a of the orbit satisfies

$$a \propto \frac{(1+q)^4}{q^2},$$

where q is the mass ratio of the two components (loser/gainer).

In a certain range of mass ratios, the radius of the Roche lobe around the loser can be approximated by

$$R_{\mathrm{L}} \approx 0.4aq^{2/9}.$$

Show that as the loser, of mass M, transfers mass to its companion, its Roche-lobe radius changes at a rate

$$\frac{d\log_e R_{\mathrm{L}}}{dt} = \alpha\frac{d\log_e M}{dt},$$

where

$$\alpha = \tfrac{20}{9}\left(q - \tfrac{4}{5}\right).$$

A star of mass M and age t has a radius R which changes in response to internal nuclear evolution and to variation in mass (provided the variation is slow), according to

$$\log_e(R/R_\odot) = \beta \log_e(M/M_\odot) + t/t_{\rm nuc},$$

where β, the slope of the ZAMS radius-mass relation, and $t_{\rm nuc}$, the nuclear timescale, can be taken as constant. As long as $R < R_{\rm L}$ the mass M remains constant but once $R > R_{\rm L}$ mass starts to flow at a rate given by

$$\frac{d\log_e(M/M_\odot)}{dt} = -\frac{1}{t_{\rm dyn}} \log \frac{R}{R_{\rm L}},$$

where $t_{\rm dyn}$ is the dynamical timescale, also constant $(t_{\rm dyn} \ll t_{\rm nuc})$.

Define $f = \log_e(R/R_{\rm L})$. As long as f is negative show that it satisfies the differential equation

$$\frac{df}{dt} = \frac{1}{t_{\rm nuc}}$$

and find a corresponding first-order linear differential equation satisfied by f when it is positive. Show that as long as $\beta > \alpha$, f tends to a small constant positive value,

$$f \to \frac{1}{\beta - \alpha} \frac{t_{\rm dyn}}{t_{\rm nuc}},$$

implying steady mass transfer on a nuclear timescale but that if $\beta < \alpha$, f grows exponentially on a dynamical timescale.

On the lower main sequence $\beta \approx 1$, while on the upper main sequence $\beta \approx 0.5$. Find the corresponding ranges of initial mass ratio q_0 for which mass transfer can proceed steadily, on a nuclear timescale once the primary has filled its Roche lobe.

5. For mass ratio q between 0.2 and 2 the ratio of Roche-lobe radius to orbital separation is given sufficiently accurately by

$$\frac{R_{\rm L}}{a} = 0.38q^{0.25}.$$

If one star transfers mass conservatively to the other in a close binary, show that the radius of the loser is a minimum when $q = \frac{7}{9}$.

The loser is a white dwarf whose radius R_* is given in terms of its mass M by

$$R_* = 0.01M^{-\frac{1}{3}}, \qquad (\dagger)$$

in solar units. Show that the Roche-lobe overflow would proceed on a rapid (i.e. hydrodynamic) timescale if $q > \frac{17}{27}$ but that it can be on a slower timescale otherwise.

A binary system with period 0.1 d consists of two white dwarfs of different mass, both in the range $0.2 - 1\,M_\odot$, for which you may assume (†) applies. Given that the Sun would fill its Roche lobe in a binary with period about 0.3 d, estimate the period at which the double-white-dwarf binary would fill its Roche lobe. Which component would reach its lobe first? What physical process or processes might bring about the necessary decrease in period within the Galactic lifetime? What would you expect to be the outcome of Roche-lobe overflow if the white dwarfs' masses are (a) $0.2 + 0.5\,M_\odot$, (b) $0.5 + 1.0\,M_\odot$ and (c) $0.6 + 0.7\,M_\odot$.

6. A binary system has components of mass M_1 and M_2 in circular orbits about their common centre of mass with a period $P = 2\pi/\Omega$. The binary separation is a. Show that the orbital angular momentum is given by

$$J = \left(\frac{M_1 M_2}{M_1 + M_2}\right) a^2 \Omega = \frac{G^{2/3} P^{1/3} M_1 M_2}{(2\pi)^{1/3} (M_1 + M_2)^{1/3}}.$$

The star of mass M_1 transfers mass to that of M_2 and loses mass to infinity through a stellar wind. The mass transfer rate is $\dot{M}_2 = -f\dot{M}_1$ so that the mass loss rate in the wind is $(1-f)\dot{M}_1$. The wind carries away a fraction λ of the specific angular momentum of the entire binary system. Show that

$$\dot{J} = \frac{\lambda(1-f) J \dot{M}_1}{M_1 + M_2}.$$

Show further that the orbital period varies as

$$P \propto M_1^{-3} M_2^{-3} (M_1 + M_2)^{3\lambda+1}.$$

The radius R of star 1 expands as a result of stellar evolution according to

$$\log_e(R/R_\odot) = n \log_e(M_1/M_\odot) + t/t_{\rm nuc},$$

where $t_{\rm nuc}$ is a constant nuclear time scale and n is constant too. The radius of the Roche lobe of star 1 may be approximated by

$$R_{\rm L} = 0.46a \left(\frac{M_1}{M_1 + M_2}\right)^{\frac{1}{3}}.$$

Stellar evolution proceeds until R slightly exceeds $R_{\rm L}$ at which point mass transfer begins at a rate

$$\dot{M}_2 = \frac{x M_1}{t_{\rm dyn}},$$

where $x = \log_e(R/R_\mathrm{L})$ and t_dyn is a the constant dynamical timescale. Deduce that

$$\frac{dx}{dt} = \frac{1}{t_\mathrm{nuc}} - \frac{x}{f t_\mathrm{dyn}} \left\{ n + \frac{5}{3} - 2fq - \left(2\lambda + \frac{2}{3}\right) \frac{(1-f)q}{1+q} \right\},$$

where $q = M_1/M_2$.

Hence find a condition on the mass ratio q that ensures that the mass transfer proceeds only on a slow nuclear time scale and indicate briefly what you would expect to happen otherwise.

7. A star, of radius R, is tidally influenced by a point mass companion at a separation $\mathbf{a}$ with $a \gg R$. This generates a fluid flow $\mathbf{v}(\mathbf{r})$ at a position $\mathbf{r}$ relative to the centre of the perturbed star that is governed by continuity

$$\frac{\partial \rho}{\partial t} + \nabla . \rho \mathbf{v} = 0,$$

where $\rho(\mathbf{r})$ is the density which must be constant on equipotential surfaces described by

$$\bar{r} = r[1 + \alpha(r) P_2(\cos\theta)].$$

The function $\alpha(r)$ is the solution to Clairaut's equation for an unperturbed star and θ is the angle between $\mathbf{a}$ and $\mathbf{r}$ which lies on the equipotential surface for which $\bar{r} = \mathrm{const}$. Show that the velocity field $\mathbf{v}$ can be written, to first order in α, as

$$\mathbf{v} = -\frac{1}{2}\beta(r)\alpha(R)\nabla K,$$

where $\beta(r)$ depends only on the structure of the unperturbed star and K is the harmonic function

$$K = \frac{\partial}{\partial t}\left\{ \frac{3}{2} r^5 a^3 l_{ij}(\mathbf{r}) l_{ij}(\mathbf{a}) \right\}.$$

Deduce that

$$\beta(r) = \frac{1}{\rho(r)} \int_R^r \frac{\alpha(r')}{\alpha(R)} \frac{d\rho}{dr'}\, dr',$$

in which the limits are chosen to ensure that β is not singular at the surface where $\rho \to 0$.

Thence show that

$$v_i = \frac{1}{Q M_1 R^2} \frac{\partial q_{ij}}{\partial t} r_j \beta(r),$$

where q_{ij} is the quadrupole tensor induced by the companion.

The rate of strain tensor

$$t_{ij} = (\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}.$$

By taking the average of the square of this over equipotential surfaces, which may now be assumed to be spherical, show that the rate of dissipation of energy is

$$
\begin{aligned}
-\frac{dE}{dt} &= \frac{1}{2}\int_V \rho\nu t_{ij}^2\, dV \\
&= \frac{9M_2^2B^2}{M_1^2R^4Q^2}\frac{1}{a^{10}}\frac{\partial \mathbf{a}}{\partial t}\left\{2\mathbf{a}\left(\mathbf{a}.\frac{\partial \mathbf{a}}{\partial t}\right) + a^2\frac{\partial \mathbf{a}}{\partial t}\right\}\int_0^{M_1} \nu\gamma(r)\, dm,
\end{aligned}
$$

where

$$\gamma = \beta^2 + \frac{2}{3}r\beta\beta' + \frac{7}{30}\beta'^2,$$

ν is the kinematic viscosity in the fluid and Q and B are properties of the unperturbed star as defined in the lectures.

Thence show that the dissipation constant σ defined in the lectures is related to the the tidal lag time τ and the viscous damping timescale defined by,

$$\frac{1}{t_{\mathrm{damp}}} = \frac{1}{M_1R^2}\int_0^{M_1} \nu\gamma\, dm,$$

by

$$\sigma = \frac{G\tau}{3B} = \frac{2}{M_1R^2Q^2}\frac{1}{t_{\mathrm{damp}}}.$$

[You may find the these isotropic integrals over a sphere of radius r,

$$\int_{|\mathbf{x}|=r} x_ix_j\, dS = \frac{4}{3}\pi r^4\delta_{ij}$$

and

$$\int_{|\mathbf{x}|=r} x_ix_jx_kx_l\, dS = \frac{4}{15}\pi r^6(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

useful.]