

Examples Sheet III

1. For the case of non-dissipative tides in a misaligned binary system show that

$$\dot{\mathbf{h}} = -\frac{M_2 B}{2\mu a b l h^2} \boldsymbol{\Omega} \cdot \mathbf{h} \boldsymbol{\Omega} \times \mathbf{h}$$

when averaged over an orbit. Deduce that both $\boldsymbol{\Omega}$ and \mathbf{h} precess about the total angular momentum vector $\mathbf{H} = \mu\mathbf{h} + I\boldsymbol{\Omega}$ and find the time to precess once around.

2. From the tidal perturbative force

$$\mathbf{f} = -\frac{9M_2^2}{M_1 R^2 (1-Q)^2 \tau_{\text{damp}}} \left(\frac{1}{R_1^{10}} \right) \{2\mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}) + r^2(\dot{\mathbf{r}} - \boldsymbol{\Omega} \times \mathbf{r})\}$$

confirm that, when $\boldsymbol{\Omega}$ is parallel to \mathbf{h} and changes are on a timescale much longer than the orbital period,

$$\dot{\mathbf{h}} = -\frac{1}{\tau_{\text{tid}}} \left\{ \frac{1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6}{(1-e^2)^{13/2}} - \frac{\Omega}{\omega} \frac{1 + 3e^2 + \frac{3}{8}e^4}{(1-e^2)^5} \right\} \mathbf{h},$$

where

$$\tau_{\text{tid}} = \frac{2}{9} \tau_{\text{damp}} \left(\frac{a}{R_1} \right)^8 \frac{M_1^2}{M_2 M} (1-Q)^2$$

and a , e and ω are the semi-major axis, eccentricity and mean angular velocity of the orbit. Show further that

$$\frac{2\dot{\omega}}{3\omega} = -\frac{\dot{a}}{a} = \frac{\dot{E}}{E} = \frac{2}{\tau_{\text{tid}}} \left\{ \frac{1 + \frac{31}{2}e^2 + \frac{255}{8}e^4 + \frac{185}{16}e^6 + \frac{25}{64}e^8}{(1-e^2)^{15/2}} - \frac{\Omega}{\omega} \frac{1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6}{(1-e^2)^6} \right\}.$$

Use conservation of angular momentum between stellar spin and orbit to confirm that

$$\frac{\dot{\Omega}}{\Omega} = -\frac{\mu h \dot{h}}{I \Omega h},$$

where I is the moment of inertia of the expanded star spinning at Ω and μh is the moment of inertia of the orbit of mean angular velocity ω . Thence deduce that when $\mu h \gg I$ the spin of the star pseudosynchronises with the orbit at periastron so that

$$\Omega = \frac{1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6}{(1 + 3e^2 + \frac{3}{8}e^4)(1-e^2)^{3/2}} \omega$$

and sketch the ratio of this to the orbital angular velocity at periastron as a function of e .

3. In an eccentric binary system star 2 is accreting from the fast wind of star 1. Recall that the accretion rate averaged over an orbit is

$$\langle \dot{M}_2 \rangle = -\frac{\dot{M}_1 (GM_2)^2}{v_w^4 a^2 \sqrt{1-e^2}},$$

where the wind speed $v_w \gg v_{\text{orb}}$, the relative orbital speed of the stars. By considering only the momentum gained by star 2 in the material it accretes from the wind show that Newton's law for the system becomes

$$\ddot{\mathbf{r}} = -\frac{GM}{r^3} \mathbf{r} + \frac{\dot{M}_2}{M_2} v_w \frac{\mathbf{r}}{r}.$$

Use this force to evaluate the changes in specific energy, angular momentum and the Laplace-Runge-Lenz vector and hence the rate of change of separation a and eccentricity e when the effects are averaged over an orbit.

4. A red giant of mass M_1 is in a binary system with a main-sequence star of mass M_2 . The red giant is losing mass in a stellar wind at a rate $\dot{M} < 0$ so that, if the intrinsic angular momentum of the stars is neglected, this wind carries off orbital angular momentum at a rate

$$\frac{\dot{J}_{\text{orb}}}{J_{\text{orb}}} = \frac{M_2 \dot{M}}{M_1 M},$$

where $M = M_1 + M_2$.

On a short timescale the radius of the giant responds according to

$$R \propto M_1^{-n} \quad 0 < n < 1$$

and the radius of its Roche lobe is approximated by

$$\frac{R_L}{a} = 0.426 \left(\frac{M_1}{M} \right)^{\frac{1}{3}}.$$

Now suppose that the giant is filling its Roche lobe ($R = R_L$) and that wind mass loss is taking place on a timescale much shorter than the nuclear timescale. Show, by differentiating $\log R/R_L$ or otherwise, that mass transfer is driven by the wind if

$$q = \frac{M_1}{M_2} < \frac{1+3n}{3(1-n)}. \quad (\dagger)$$

What happens otherwise?

Show further that, when (†), is satisfied, the rate of mass transfer to the main-sequence star

$$\dot{M}_2 = -\frac{1 + 3n - 3(1 - n)q}{(1 + q)(5 - 3n - 6q)}\dot{M}.$$

What is the physical consequence if $6q > 5 - 3n$?

5. The red star in a cataclysmic variable has a surface magnetic field B_* . Outside the star this falls off like a dipole so that

$$\frac{B}{B_*} = \left(\frac{R}{R_*}\right)^{-3}$$

at a distance R from the centre of the star which has radius R_* . Approximating the wind of this star as an equatorial outflow at constant speed v show that the Alfvén radius R_A is given by

$$\left(\frac{R_A}{R_*}\right)^4 = \frac{4\pi B_*^2 R_*^2}{\dot{M} v \mu_0},$$

where \dot{M} is the rate of mass loss in the stellar wind.

The radius of the donor star of mass M_2 behaves as

$$\frac{R_*}{R_\odot} = \left(\frac{M_2}{M_\odot}\right)^{\frac{13}{15}} \quad \text{and} \quad \frac{R_L}{a} = 0.462 \left(\frac{M_2}{M}\right)^{\frac{1}{3}},$$

where a is the separation of the circular orbit and M is the total mass of the system. Suppose that magnetic braking is the only source of angular momentum loss and $-\dot{M} \ll -\dot{M}_2 \approx \dot{M}_1$ to show that the rate of steady mass transfer is

$$\dot{M}_2 = \frac{15q}{19 - 15q} \frac{M}{M_1} \left(\frac{R_A}{a}\right)^2 \dot{M} \propto \dot{M}^{1/2},$$

where $q = M_2/M_1$.

For a system with $M_1 = 0.6 M_\odot$ and $M_2 = 0.3 M_\odot$ in which v is the escape velocity from star 2 and $B_* = 10^3 \text{ G} = 0.1 \text{ T}$, what mass-loss rate \dot{M} is required to maintain a mass-transfer rate of $\dot{M}_2 = 10^{-10} M_\odot \text{ yr}^{-1}$? What is R_A/R_* in this case?

6. A supergiant of C/O core mass M_c and envelope mass M_{env} , of which the binding energy may be expressed as

$$E_{\text{bind}} \approx -\frac{2GM_{\text{env}}^2}{R_G},$$

where R_G is a fiducial radius defined by

$$\frac{R_G}{R_\odot} \approx 1,000 \left(\frac{M_c}{M_\odot}\right)^2 \left(\frac{M_{\text{env}}}{M_\odot}\right)^{-\frac{1}{3}},$$

is in a binary with a C/O white dwarf of mass M_{wd} .

The giant fills its Roche lobe and dynamical mass transfer leading to common-envelope evolution ensues. Show that, if the common envelope efficiency is α_{ce} , the final separation of the cores when the envelope has been lost is a_f where

$$\frac{a_f}{R_G} \approx \frac{\alpha_{\text{ce}} M_c M_{\text{wd}}}{2 M_{\text{env}}^2},$$

in the limit $a_f \ll a_i$, where a_i is the initial separation.

The radius of a white dwarf of mass M_{wd} can be approximated by

$$\frac{R_{\text{wd}}}{R_{\odot}} \approx 0.01 \left(\frac{M_{\odot}}{M_{\text{wd}}} \right)^{\frac{1}{3}}$$

and the radius of the hot giant core by

$$R_c(M_c) \approx 5R_{\text{wd}}(M_c)$$

The spiralling cores coalesce if $a_f \leq 3 \max(R_{\text{wd}}, R_c)$. Estimate the minimum envelope mass M_{crit} required for the cores to coalesce if $M_c = 0.6 M_{\odot}$, $M_{\text{wd}} = 0.9 M_{\odot}$ and $\alpha_{\text{ce}} = 1.0$.

Suppose the process of coalescence heats the degenerate white dwarf and the supergiant core to a temperature at which carbon burning $^{12}\text{C} + ^{12}\text{C} \rightarrow ^{24}\text{Mg}$ (13.93 MeV per reaction) can ignite. Estimate the total nuclear energy that can be released and compare it with the binding energy of the white dwarf which may be modelled as an $n = 3/2$ polytrope.

Comment on the result for M_{env} in the range from well below to well above M_{crit} .

7. A binary system consists of two stars of masses M_1 and M_2 in an orbit with eccentricity e and semi-major axis a . At a point on the orbit when the separation is r , star 1 loses a fraction $1 - f$ of the total mass of the system $M = M_1 + M_2$ in an isotropic supernova explosion over a time short compared with the orbital period. After the explosion the total mass is $M' = fM$. Show that the system is most easily unbound if the explosion takes place at periastron in which case it unbinds if

$$f < \frac{1+e}{2}$$

and hence that such a supernova expelling half the mass in a system originally in a circular orbit unbinds the system.