Astrophysical Fluid Dynamics
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Mathematical Tripos, Part III
Michaelmas Term 2011

Example Sheet 1

1. *Validity of a fluid approach*

The Coulomb cross-section for 'collisions' (i.e. large-angle scatterings) between electrons and protons is $\sigma \approx 1 \times 10^{-4}(T/\mathrm{K})^{-2}\,\mathrm{cm}^2$. Why does it depend on the inverse square of the temperature?

Using the numbers quoted in lectures, estimate the order of magnitude of the mean free path and collision frequency in (i) the centre of the Sun, (ii) the solar corona, (iii) a molecular cloud and (iv) the hot phase of the interstellar medium. Comment on your answers.

2. *Vorticity equation*

Show that the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ of an ideal fluid without a magnetic field satisfies the equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \boldsymbol{\nabla} p \times \boldsymbol{\nabla} v\,,$$

where $v = 1/\rho$ is the specific volume. Explain why the last term, which acts as a source of vorticity, can also be written as $\boldsymbol{\nabla} T \times \boldsymbol{\nabla} S$. Under what conditions does the source term vanish, and in what senses can the vorticity then be said to be conserved?

Show that the *potential vorticity*

$$\frac{1}{\rho}\boldsymbol{\omega} \cdot \boldsymbol{\nabla} S$$

is conserved (as a pseudoscalar quantity) under more general circumstances.

3. *Validity of the ideal MHD equations*

Three effects neglected in a non-relativistic theory of MHD are (i) the displacement current in Maxwell's equations (compared to the electric current), (ii) the bulk electrostatic force on the fluid (compared to the magnetic Lorentz force) and (iii) the electrostatic energy (compared to the magnetic energy). Verify the self-consistency of these approximations by using order-of-magnitude estimates to show that these three neglected effects are all smaller than those retained by a factor of order U^2/c^2. (You may wish to consult Jackson, *Classical Electrodynamics*, or a similar book.)

4. *Magnetostatic equilibrium 1*

A simple model for a filament in the solar atmosphere involves a two-dimensional magnetostatic equilibrium in the (x,z) plane with uniform gravity $\boldsymbol{g} = -g\,\boldsymbol{e}_z$. The gas is isothermal with isothermal sound speed c_{s}. The density and magnetic field depend only on x and the field lines become straight as $|x| \to \infty$.

Show that the solution is of the form

$$B_z = B_0 \tanh(kx)\,,$$

where k is a constant to be determined. Sketch the field lines and find the density distribution.

5. *Magnetostatic equilibrium 2*

(a) Referred to cylindrical polar co-ordinates (r, ϕ, z) a purely azimuthal magnetic field $\mathbf{B} = (0, B, 0)$. Show that

$$(\mathrm{curl}\,\mathbf{B}) \times \mathbf{B} = -(B/r)\nabla(rB)$$

and hence that a necessary condition for the force exerted by the field to be balanced by pressure gradients alone is that $\partial B/\partial z = 0$.

(b) Suppose that the gas in a star satisfies a barotropic equation of state (ie that the pressure is a function of the density ρ) and that the forces exerted by the magnetic field are balanced by pressure gradients and gravity. Show that the azimuthal field must have the form

$$B = r^{-1} f(r^2\rho),$$

where $f(x)$ is an arbitrary function.

Consider the simplest case, when $f(x) = \lambda x$ with λ a constant, and show that $\lambda < \lambda_0 \sim (\mu_0 G)^{\frac{1}{2}}$, where G is the gravitational constant.

6. *Helicity*

The magnetic helicity in a volume V is

$$H_{\mathrm{m}} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V \,.$$

A *thin, untwisted magnetic flux tube* is a thin tubular structure consisting of the neighbourhood of a smooth curve C, such that the magnetic field is confined within the tube and is parallel to C.

(a) Consider a simple example of a single, closed, untwisted magnetic flux tube such that

$$\boldsymbol{B} = B(R, z) \, \boldsymbol{e}_\phi \,,$$

where (R, ϕ, z) are cylindrical polar coordinates and $B(R, z)$ is a positive function localized near $(R = a, z = 0)$. The tube is contained entirely within V. Show that the magnetic helicity of this field is uniquely defined and equal to zero.

(b) Use the fact that H_{m} is conserved in ideal MHD to argue that the magnetic helicity of any single, closed, untwisted and unknotted flux tube contained within V is also zero.

(c) Consider a situation in which V contains two such flux tubes T_1 and T_2. Let F_1 and F_2 be the magnetic fluxes associated with T_1 and T_2. By writing $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$, etc., and assuming that the tubes are thin, show that

$$H_{\mathrm{m}} = \pm 2 F_1 F_2$$

if the tubes are simply interlinked, while $H_{\mathrm{m}} = 0$ if they are unlinked.

7. *Variational principles*

The magnetic energy in a volume V bounded by a surface S is

$$E_{\mathrm{m}} = \int_V \frac{B^2}{2\mu_0} \, \mathrm{d}V \,.$$

(a) Making use of the representation $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ of the magnetic field in terms of a magnetic vector potential, show that the magnetic field that minimizes E_{m}, subject to the tangential components of $\boldsymbol{A}$ being specified on S, is a potential field. Argue that this constraint corresponds to specifying the normal component of $\boldsymbol{B}$ on S.

(b) Making use of the representation $\boldsymbol{B} = \boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\beta$ of the magnetic field in terms of Euler potentials, show that the magnetic field that minimizes E_{m}, subject to α and β being specified on S, is a force-free field. Argue that this constraint corresponds to specifying the normal component of $\boldsymbol{B}$ on S and also the connection of points on S by magnetic field lines.

Please send any comments and corrections to `jcbp2@damtp.cam.ac.uk`