## Example Sheet 2

1. *Friedrichs diagrams*

The dispersion relations $\omega(\boldsymbol{k})$ for Alfvén and magnetoacoustic waves in a uniform medium are given by

$$v_\mathrm{p}^2 = v_\mathrm{a}^2 \cos^2\theta\,,$$

$$v_\mathrm{p}^4 - (c^2 + v_\mathrm{a}^2)v_\mathrm{p}^2 + c^2 v_\mathrm{a}^2 \cos^2\theta = 0\,,$$

where $v_\mathrm{p} = \omega/k$ is the phase velocity and $\theta$ is the angle between $\boldsymbol{k}$ and $\boldsymbol{B}$. Use the form of $v_\mathrm{p}(\theta)$ for each mode to calculate the group velocities $\boldsymbol{v}_\mathrm{g} = \partial\omega/\partial\boldsymbol{k}$, determining their components parallel and perpendicular to $\boldsymbol{B}$.

Sketch the phase and group diagrams by tracking $\boldsymbol{v}_\mathrm{p} = v_\mathrm{p}\hat{\boldsymbol{k}}$ and $\boldsymbol{v}_\mathrm{g}$, respectively, over the full range of $\theta$. Treat the cases $c > v_\mathrm{a}$ and $c < v_\mathrm{a}$ separately. By analysing the limit $\theta \to \pi/2$, show that the group diagram for the slow wave has a cusp at speed $cv_\mathrm{a}(c^2 + v_\mathrm{a}^2)^{-1/2}$.

2. *Shock relations*

The Rankine–Hugoniot relations in the rest frame of a non-magnetic shock are

$$[\rho u_x]_1^2 = 0\,,$$

$$[\rho u_x^2 + p]_1^2 = 0\,,$$

$$[\rho u_x(\tfrac{1}{2}u_x^2 + w)]_1^2 = 0\,,$$

where $[Q]_1^2 = Q_2 - Q_1$ is the difference between downstream and upstream values of any quantity $Q$. Solve these equations for a polytropic gas to obtain

$$\frac{\rho_2}{\rho_1} = \frac{u_{x1}}{u_{x2}} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2}\,,$$

$$\frac{p_2}{p_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{(\gamma+1)}\,,$$

$$\mathcal{M}_2^2 = \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{2\gamma\mathcal{M}_1^2 - (\gamma-1)}\,,$$

where $\mathcal{M} = u_x/v_\mathrm{s}$ is the Mach number.

Show that the last relation can be written

$$X_1 X_2 = 1$$

in terms of the auxiliary variable

$$X = \frac{2\gamma}{\gamma+1}(\mathcal{M}^2 - 1) + 1\,.$$

Express $\rho_2/\rho_1$ and $p_2/p_1$ in terms of $X_1$. What is the acceptable range of $X_1$?

Show that the entropy change through the shock is given by

$$\frac{[s]_1^2}{c_v} = \ln X_1 - \gamma \ln\left[\frac{(\gamma+1)X_1 + (\gamma-1)}{(\gamma-1)X_1 + (\gamma+1)}\right]$$

and deduce that only compression shocks ($\rho_2 > \rho_1$) are physically realizable.

3. *One-dimensional similarity flow*

(a) A homentropic, polytropic gas flows in one dimension. Determine all types of solution of the form

$$u = c\,\tilde{u}(\xi)\,,$$

$$v_{\mathrm{s}} = c\,\tilde{v}(\xi)\,,$$

where $c$ is a constant velocity and $\xi = x/ct$. (The type of solution in which $\tilde{u}$ and $\tilde{v}$ depend linearly on $\xi$ is called a *rarefaction wave.*) Show that at least one set of characteristics consists of a family of straight lines.

(b) The gas is initially at rest in a long, thin tube ($x > 0$) contained by a piston ($x = 0$). The gas is uniform and has adiabatic sound speed $c$. At time $t = 0$ the piston starts to move in the $+x$ direction with uniform speed $\mathcal{M}c$. Explain why the subsequent solution has the above similarity form. Determine the solution appropriate to this problem. *Hint: the solution includes a shock but no rarefaction wave.*

(c) As above, except that the piston moves in the $-x$ direction with uniform speed $\mathcal{M}c$. *Hint: the solution includes a rarefaction wave but no shock.* What happens if $\mathcal{M} > 2/(\gamma-1)$?

4. *Waves on nonuniform fields*

Consider small disturbances $\mathbf{b}(x,z,t), \mathbf{u}(x,z,t)$ to a constant-current magnetic field $\mathbf{B} = (\mu_0 Jz, 0, 0)$. Assume that $\mathbf{b}, \mathbf{u}$ lie in the $(x,z)$ plane so that $\mathbf{u} = \nabla \times (0,\psi,0)$ (so that $\nabla\cdot\mathbf{u} = 0$), and $\mathbf{b} = \nabla\times(0,A,0)$, and that there is no applied electric field in the $y$-direction. Both $u_z$ and $b_z$ vanish at $z = a$, $z = 2a$.

Defining the perturbation current $j = -\mu_0^{-1}\nabla^2 A$ and vorticity $\omega = -\nabla^2\psi$, show that the following linearized equations hold:

$$\frac{\partial A}{\partial t} = Jz\frac{\partial \psi}{\partial x}, \quad \frac{\partial \omega}{\partial t} = (\rho\mu_0)^{-1} Jz \frac{\partial j}{\partial x} \qquad (1)$$

Show that solutions can be found in the form $\psi = \hat{\psi}(z)e^{ikx+i\sigma t}$ etc., and write down the equation satisfied by $\hat{\psi}$. Use this to show that the system is stable. Indicate, giving reasons, whether you would expect to be able to find allowed real values of $\sigma$.

5. *Bondi Accretion*

Write down the equations of steady, spherical accretion of a polytropic gas in an arbitrary gravitational potential $\Phi(r)$.

Accretion on to a black hole can be approximated within a Newtonian theory by using the *Paczyński–Wiita potential*

$$\Phi = -\frac{GM}{r - r_\mathrm{h}} \,,$$

where $r_\mathrm{h} = 2GM/c^2$ is the radius of the event horizon and $c$ is the speed of light.

Show that the sonic radius $r_\mathrm{s}$ is related to $r_\mathrm{h}$ and the nominal accretion radius $r_\mathrm{a} = GM/2v_\mathrm{s0}^2$ (where $v_\mathrm{s0}$ is the sound speed at infinity) by

$$2r_\mathrm{s}^2 - [(5 - 3\gamma)r_\mathrm{a} + 4r_\mathrm{h}]\, r_\mathrm{s} + 2r_\mathrm{h}^2 - 4(\gamma - 1)r_\mathrm{a}r_\mathrm{h} = 0 \,.$$

Argue that the accretion flow passes through a unique sonic point for any value of $\gamma > 1$. Assuming that $v_\mathrm{s0} \ll c$, find approximations for $r_\mathrm{s}$ in the cases (i) $\gamma < 5/3$, (ii) $\gamma = 5/3$ and (iii) $\gamma > 5/3$.

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