Example Sheet 3

1. *Critical points of magnetized outflows*

The integrals of the equations of ideal MHD for a steady axisymmetric outflow are

$$\boldsymbol{u} = \frac{k\boldsymbol{B}}{\rho} + R\omega\,\boldsymbol{e}_\phi\,, \tag{1}$$

$$u_\phi - \frac{B_\phi}{\mu_0 k} = \frac{\ell}{r}\,, \tag{2}$$

$$s = s(\psi)\,, \tag{3}$$

$$\tfrac{1}{2}|\boldsymbol{u} - R\omega\,\boldsymbol{e}_\phi|^2 + w + \Phi - \tfrac{1}{2}R^2\omega^2 = \varepsilon'\,, \tag{4}$$

where $k(\psi)$, $\omega(\psi)$, $\ell(\psi)$, $s(\psi)$ and $\varepsilon'(\psi)$ are surface functions. Assume that the magnetic flux function $\psi(R,z)$ is known from a solution of the Grad–Shafranov equation, and let the cylindrical radius R be used as a parameter along each magnetic field line. Then the poloidal magnetic field $\boldsymbol{B}_\mathrm{p} = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\phi$ is a known function of r on each field line. Assume further that the surface functions $k(\psi)$, $\omega(\psi)$, $\ell(\psi)$, $s(\psi)$ and $\varepsilon'(\psi)$ are known.

Show that equations (1)–(3) can then be used, in principle, and together with the equation of state, to determine the velocity $\boldsymbol{u}$ and the specific enthalpy w as functions of ρ and R on each field line. Deduce that equation (4) has the form

$$f(\rho, R) = \varepsilon' = \text{constant}$$

on each field line.

Show that

$$-\rho\frac{\partial f}{\partial \rho} = \frac{u_\mathrm{p}^4 - (c^2 + v_\mathrm{a}^2)u_\mathrm{p}^2 + c^2 v_\mathrm{ap}^2}{u_\mathrm{p}^2 - v_\mathrm{ap}^2}\,,$$

where v_s is the adiabatic sound speed, v_a is the (total) Alfvén speed and the subscript ‘p’ denotes the poloidal (meridional) component. Deduce that the flow has critical points where u_p equals the phase speed of axisymmetric fast or slow magnetoacoustic waves. What condition must be satisfied by $\partial f/\partial r$ for the flow to pass through these critical points?

2. *Radial oscillations of a star*

Show that purely radial (i.e. spherically symmetric) oscillations of a spherical star satisfy the Sturm–Liouville equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\gamma p}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\xi_r)\right] - \frac{4}{r}\frac{\mathrm{d}p}{\mathrm{d}r}\xi_r + \rho\omega^2\xi_r = 0\,.$$

How should ξ_r behave near the centre of the star and near the surface $r = R$ at which $p = 0$?

Show that the associated variational principle can be written in the equivalent forms

$$\begin{aligned}\omega^2\int_0^R \rho|\xi_r|^2\,r^2\,\mathrm{d}r &= \int_0^R\left[\frac{\gamma p}{r^2}\left|\frac{\mathrm{d}}{\mathrm{d}r}(r^2\xi_r)\right|^2 + 4r\frac{\mathrm{d}p}{\mathrm{d}r}|\xi_r|^2\right]\mathrm{d}r \\ &= \int_0^R\left[\gamma p r^4\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\xi_r}{r}\right)\right|^2 + (4-3\gamma)r\frac{\mathrm{d}p}{\mathrm{d}r}|\xi_r|^2\right]\mathrm{d}r\,,\end{aligned}$$

where γ is assumed to be independent of r. Deduce that the star is unstable to purely radial perturbations if and only if $\gamma < 4/3$. Why does it not follow from the first form of the variational principle that the star is unstable for all values of γ?

Can you reach the same conclusion using only the virial theorem?

3. *Waves in an isothermal atmosphere*

Show that linear waves of frequency ω and horizontal wavenumber k_h in a plane-parallel isothermal atmosphere satisfy the equation

$$\frac{\mathrm{d}^2\xi_z}{\mathrm{d}z^2} - \frac{1}{H}\frac{\mathrm{d}\xi_z}{\mathrm{d}z} + \frac{(\gamma-1)}{\gamma^2H^2}\xi_z + (\omega^2 - N^2)\left(\frac{1}{v_\mathrm{s}^2} - \frac{k_\mathrm{h}^2}{\omega^2}\right)\xi_z = 0\,,$$

where H is the isothermal scale-height, N is the Brunt–Väisälä frequency and v_s is the adiabatic sound speed.

Consider solutions of the vertically wavelike form

$$\xi_z \propto \mathrm{e}^{z/2H}\exp(\mathrm{i}k_z z)\,,$$

where k_z is real, so that the wave energy density (proportional to $\rho|\boldsymbol{\xi}|^2$) is independent of z. Obtain the dispersion relation connecting ω and $\boldsymbol{k}$. Assuming that $N^2 > 0$, show that propagating waves exist in the limits of high and low frequencies, for which

$$\omega^2 \approx v_\mathrm{s}^2k^2 \quad \text{(acoustic waves)} \qquad \text{and} \qquad \omega^2 \approx \frac{N^2k_\mathrm{h}^2}{k^2} \quad \text{(gravity waves)}$$

respectively. Show that the minimum frequency at which acoustic waves propagate is $v_\mathrm{s}/2H$.

4. *Waves in a rotating fluid*

Write down the equations of ideal gas dynamics in cylindrical polar coordinates (R, ϕ, z). Consider a steady, axisymmetric basic state in uniform rotation, with density $\rho(R, z)$, pressure $p(R, z)$ and velocity $\boldsymbol{u} = R\Omega\,\boldsymbol{e}_\phi$. Determine the linearized equations governing axisymmetric perturbations of the form

$$\mathrm{Re}\left[\rho'(R, z)\,\mathrm{e}^{-\mathrm{i}\omega t}\right] ,$$

etc. If the basic state is adiabatically stratified (i.e. $s =$ constant) and self-gravity may be neglected, show that the linearized equations reduce to

$$-\mathrm{i}\omega u'_R - 2\Omega u'_\phi = -\frac{\partial W}{\partial R} ,$$

$$-\mathrm{i}\omega u'_\phi + 2\Omega u'_R = 0 ,$$

$$-\mathrm{i}\omega u'_z = -\frac{\partial W}{\partial z} ,$$

$$-\mathrm{i}\omega W + \frac{v_\mathrm{s}^2}{\rho}\left[\frac{1}{R}\frac{\partial}{\partial R}(R\rho u'_R) + \frac{\partial}{\partial z}(\rho u'_z)\right] = 0 ,$$

where $W = p'/\rho$.

Eliminate $\boldsymbol{u}'$ to obtain a second-order partial differential equation for W. Is the equation of elliptic or hyperbolic type? What are the solutions of this equation if the fluid has uniform density and fills a cylindrical container $\{R < a,\ 0 < z < H\}$ with rigid boundaries?

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