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Structure and Evolution of Stars

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[For Examples class 2-4pm Monday 22th November] Michaelmas Term, 2010

Examples Sheet III

1. In solar-like stars nuclear burning is dominated by the ppI and ppII chains

$$
\begin{array}{l}
{}^{1}\mathrm{H}({}^{1}\mathrm{H},\mathrm{e}^{+}\nu){}^{2}\mathrm{H}({}^{1}\mathrm{H},\gamma){}^{3}\mathrm{He}({}^{3}\mathrm{He},2{}^{1}\mathrm{H}){}^{4}\mathrm{He}\\
\qquad\qquad\qquad\qquad\quad |\\
\qquad\qquad\qquad\qquad {}^{3}\mathrm{He}({}^{4}\mathrm{He},\gamma){}^{7}\mathrm{Be}(\mathrm{e}^{-},\nu){}^{7}\mathrm{Li}({}^{1}\mathrm{H},{}^{4}\mathrm{He}){}^{4}\mathrm{He}.
\end{array}
$$

The reaction rate between species i and j is

$$\frac{\lambda_{ij} n_i n_j}{1+\delta_{ij}}, \qquad (*)$$

where n_i is the number density of species i, δ_{ij} is the Kronecker delta and $\lambda_{ij} \propto \eta^2 \mathrm{e}^{-\eta}$, where $\eta = 42.48(AZ_i^2 Z_j^2 T_6^{-1})^{1/3}$, $A = A_i A_j/(A_i + A_j)$ is the reduced atomic mass of the two reacting nuclei, Z_i is the atomic number of species i and the T_6 is related to temperature T by $T_6 = T/10^6\,\mathrm{K}$.

The beta decay of ${}^{7}\mathrm{Be}$ is fast compared to all other reactions so that ${}^{7}\mathrm{Li}$ is the predominant species of atomic mass 7 and all major species can be identified by $i \approx A_i$. Show that the temperature dependence of the rate r_{11} at the centre of the Sun, where $T_6 \approx 15$, of the reaction ${}^{1}\mathrm{H}({}^{1}\mathrm{H},\mathrm{e}^{+}\nu){}^{2}\mathrm{H}$ can be written as $r_{11} \propto T^{\alpha}$, where $\alpha = \frac{1}{3}(\eta - 2) \approx 4$. Also show that β and γ are approximately 16 (with $\gamma > \beta$) in the expressions $r_{33} \propto T^{\beta}$ and $r_{34} \propto T^{\gamma}$.

Show that the rate of change of protons obeys

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = -\lambda_{11} n_1^2 - \lambda_{21} n_2 n_1 + \lambda_{33} n_3^2 - \lambda_{17} n_1 n_7,$$

and obtain the equivalent equations for n_2, n_3 and n_4.

At the centre of the Sun the characteristic timescale of r_{11} is about 10^{10} yr while that of r_{12} is about 1 s. The characteristic timescale for n_3 to reach equilibrium is $\tau \approx 6 \times 10^5$ yr. By making an appropriate approximation, to be explained, show that

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} \approx -\frac{3}{2}\lambda_{11} n_1^2 + \lambda_{33} n_3^2 - \lambda_{17} n_1 n_7$$

and

$$\frac{\mathrm{d}n_3}{\mathrm{d}t} \approx \frac{1}{2}\lambda_{11} n_1^2 - \lambda_{33} n_3^2 - \lambda_{34} n_3 n_4$$

near the centre of the Sun.

Show further that $n_3 \approx n_{3\mathrm{e}}$ where

$$n_{3\mathrm{e}} = -\frac{\lambda_{34}n_4}{2\lambda_{33}} + \sqrt{\left(\frac{\lambda_{34}n_4}{2\lambda_{33}}\right)^2 + \frac{\lambda_{11}n_1^2}{2\lambda_{33}}}.$$

Consider a small perturbation of the form $n_3 = n_{3\mathrm{e}} + x$ about this equilibrium and linearise the evolution equation for n_3 to obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{\tau},$$

where $\tau = (2\lambda_{33}n_{3\mathrm{e}} + \lambda_{34}n_4)^{-1}$.

Estimate the temperature at which τ is comparable to the age of the Sun.

Sketch the abundances X_1 and X_3 of ^{1}H and ^{3}He as a function of radius in the Sun today.

2. A set of fully radiative stars has uniform mean molecular weight μ, constant opacity and energy generation rate given by $\epsilon = \epsilon_0 \rho T^{16}$ where ϵ_0 is constant. Radiation pressure is negligible. Use a simple homology argument to show that for the set of stars

$$L \text{ varies as } \mu^4 M^3,$$
$$T_\mathrm{c} \text{ varies as } \mu^{7/19} M^{4/19},$$
$$\text{and } R \text{ varies as } \mu^{12/19} M^{15/19},$$

where M is the stellar mass, L the stellar luminosity, R the stellar radius and T_c the central temperature.

Hence show that the slope of the theoretical main sequence for such a set of stars is

$$\frac{d \log L}{d \log T_\mathrm{eff}} = \frac{76}{9},$$

where T_eff is the effective (surface) temperature.

Consider now two such sets of stars which differ in that one set is composed of pure hydrogen while the other is of pure helium. By considering the ratio of luminosities at fixed effective temperature for the two sets of stars, or otherwise, show that the helium main sequence lies below and to the left of the hydrogen main sequence in the Hertzsprung–Russell diagram.

3. Indicate why in a fully convective star the equation of state may be taken to be $P = KT^{5/2}$ where K is a constant. Integrations for the atmospheric structure show that

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$K = Ag^{\nu}T_{\rm e}^{-\lambda}$ where A, ν and λ are constant, g is the surface gravity and $T_{\rm e}$ the effective temperature. Derive a luminosity–mass–radius relation in the form

$$\frac{L}{4\pi\sigma R^2} = CR^{\alpha}M^{\beta} \ ,$$

where C, α and β are constant and α and β depend solely on ν and λ. Show that, when $\nu = 3/4$, $T_{\rm e}$ is constant. In this case show that the time for a fully convective star to contract to radius R_s radiating its gravitational energy is

$$t = \frac{GM^2}{7\left(4\pi R_s^3 T_{\rm e}^4\right)\sigma}$$

[*You may quote any properties of polytropes that you need.*]

4. A white dwarf may be approximated by a two-zone model. A helium interior is composed of a non-relativistic fully degenerate electron gas at constant temperature $T_{\rm c}$ with equation of state

$$P_{\rm e} = K\rho^{5/3}, \qquad (*)$$

where K is constant and $P_{\rm e}$ is the electron pressure. The very thin outer layers are composed of hydrogen gas in radiative equilibrium obeying the perfect-gas equation of state with negligible radiation pressure and with opacity given by Kramers' law in the form

$$\kappa = \kappa_0 \rho T^{-3.5}.$$

The transition between the inner and outer zones is defined to be where the total pressure given by the perfect-gas law identically equals the electron pressure given by equation $(*)$. Show that:

(i) in the very thin outer layers of the white dwarf

$$P^2 = \frac{64}{51}\frac{\pi acGM}{\kappa_0 L}\left(\frac{\mathcal{R}}{\mu}\right)T^{8.5} \equiv JT^{8.5},$$

where M and L are the mass and luminosity of the white dwarf;

(ii) the temperature at the transition $T_{\rm tr}$ is

$$T_{\rm tr} = \left(\frac{\mathcal{R}}{\mu}\right)^{10/7} K^{-6/7}J^{-2/7};$$

(iii) the luminosity of the white dwarf is

$$L = \frac{64}{51}\frac{\pi acGM}{\kappa_0}\left(\frac{\mu}{\mathcal{R}}\right)^4 K^3\, T_{\rm c}^{3.5}.$$

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By taking plausible numerical values, which should be stated, estimate to order of magnitude the temperature in the interior of the white dwarf.

Comment on the source of energy for the white dwarf's luminosity and estimate, in years, to order of magnitude, the cooling time scale of the white dwarf.

[$K \approx 10^{13}\,\mathrm{dyne\,cm^3 g^{-5/3}}$, $\kappa_0 \approx 4 \times 10^{24}\,\mathrm{cm^5 g^{-2} K^{7/2}}$.]

5. A supernova light curve can be modelled by using the conservation of energy,

$$\dot{E} + P\dot{V} = \epsilon - \frac{\partial L}{\partial m},$$

where E is the total energy per unit mass, P is the pressure, V is the volume per unit mass, ϵ is the heating term and $\frac{\partial L}{\partial m}$ is the radiative loss from diffusion. Assume there is no heating and further that the gas is dominated by radiation so that $E = aT^4V$ and $P = \frac{1}{3}aT^4$. Also approximate the radiative losses by $\frac{\partial L}{\partial m} \approx \frac{L}{M} \approx \frac{E}{\tau_{\mathrm{diff}}}$, where τ_{diff} is the timescale for radiative diffusion. We approximate τ_{diff} by $\frac{f\kappa M}{cR}$, where f is a constant, κ is the opacity, M the pre-supernova mass of the star, c the speed of light and $R(t)$ the outer radius of the expanding gas. If this ejected gas expands homologously show that

$$\frac{d}{dt}\log_e(T^4R^4) = -\frac{1}{\tau_{\mathrm{diff}}}.$$

Suppose the supernova expands at constant velocity v so that $R(t) = R_0 + vt$ and assume that κ remains constant. Show that

$$L(t) = L_0 \exp\left(-\frac{\tau_{\mathrm{h}}t + \frac{1}{2}t^2}{\tau_{\mathrm{h}}\tau_{\mathrm{diff},0}}\right),$$

where the hydrodynamic timescale $\tau_{\mathrm{h}} = \frac{R_0}{v}$ and find an expression for L_0 in terms of T_0 and R_0.

How is the supernova light curve affected if either the opacity is lowered, the star is more massive or the star is large.

6. The late time luminosity of a supernova is powered by the decay of ^{56}Co to ^{56}Fe. The half life of this decay is 77.1 d and the energy per unit mass generated by the reaction is $6.4 \times 10^{16}\,\mathrm{erg\,g^{-1}}$. The initial luminosity generated by ^{56}Co in supernova 1987A was $10^{42}\,\mathrm{erg\,s^{-1}}$. What was the total energy released in the decay of the ^{56}Co? Given that ^{56}Co was formed from the decay of ^{56}Ni, how many solar masses of ^{56}Ni were synthesized in the supernova?

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