

Examples Sheet IV

1. Consider a cluster of chemically homogeneous massive stars. The stellar material is an ideal gas and radiation pressure is negligible. The energy generation, opacity and whether the stellar interiors are convective or radiative depends on a star's mass. There are three mass-ranges of behaviour over the full main sequence.

(i) For $M/M_{\odot} > 2$, the stars are fully radiative, energy generation is by the CNO cycle and opacity dominated by electron-scattering, such that $\epsilon = \epsilon_0 X \rho T^{13}$ and $\kappa = \kappa_0$.

(ii) For $0.5 < M/M_{\odot} < 2$, the stars are also fully radiative, energy generation is by the p-p chain and opacity determined by Kramers' opacity, such that $\epsilon = \epsilon_0 X \rho T^7$ and $\kappa = \kappa_0 \rho T^{-7/2}$.

(iii) For $M < 0.5M_{\odot}$, the stars are fully convective, energy generation is by the p-p chain and opacity dominated by H^- anions, such that $\epsilon = \epsilon_0 X \rho T^3$ and $\kappa = \kappa_0 \rho^{1/2} T^9$.

Here κ_0 and ϵ_0 are constants. Using homology show for the three different types of main-sequence star that $L \propto M^{\alpha}$ with $\alpha = 3$, $5\frac{6}{19}$ and $1\frac{47}{66}$ respectively. Then determine the gradient that each main sequence makes in the theoretical Hertzsprung-Russell diagram and sketch this diagram showing the three main-sequences and how they combine.

For case (iii), when the stars are convective, show for a star of fixed mass that $L \propto X^{17/66} \mu^{103/66}$, assuming that $Z = 0$ and initially $X = 1$. Using this result estimate how such a star evolves away from the main-sequence as it burns hydrogen to helium. Sketch an example track onto your HR diagram. Why can this analysis not be performed for the stars that have radiative interiors?

2. A red giant can be modelled by an isothermal degenerate helium core surrounded by a thin hydrogen burning shell, above which is a radiative region which is itself surrounded by a deep convective envelope. The core is of mass M_1 and radius R_1 related by

$$M_1^{1/3} R_1 = A = \text{const.}$$

At the base of the radiative region just above the core boundary is the thin hydrogen-burning shell which generates the entire luminosity L . The entire radiative envelope has a negligible mass while the convective envelope above it has a significant amount of mass. The opacity obeys $\kappa = \kappa_0 \rho^n / T^m$, with n and m constant. Show that, when radiation pressure is neglected, the relation between P and T in the radiative region is

$$P = C (T^{4+m+n} + T_0^{4+m+n})^{1/(n+1)},$$

where

$$C = \left[\frac{16\pi acGM_1}{3\kappa_0 L} \left(\frac{R}{\mu} \right)^n \frac{(n+1)}{(n+m+4)} \right]^{1/(n+1)}$$

and T_0 is an appropriate constant of integration.

The temperature at the boundary between the radiative zone and the convective envelope is T_b . Show that

$$T_0 = T_b \left\{ \left(\frac{\gamma-1}{\gamma} \right) \left(\frac{4+m+n}{n+1} \right) - 1 \right\}^{+1/(4+n+m)},$$

where γ is the ratio of specific heats throughout the convective zone.

Show that, in regions near the shell, well below the inner boundary of the convective envelope where $T \gg T_b$ and hence $T \gg T_0$, the dependence of temperature on radius r is approximately

$$T = \frac{\mu}{R} \frac{GM_1(n+1)}{(4+n+m)r}.$$

Use this to show that, when $n = 1$, $m = 3$ and the energy generation rate is given by $\epsilon = \epsilon_0 \rho T^{10}$, with $\epsilon_0 = \text{const}$,

$$L = \frac{4\pi}{13} C^2 \epsilon_0 \left(\frac{\mu}{R} \right)^2 \left(\frac{\mu GM_1}{4R} \right)^{16} \frac{1}{R_1^{13}}.$$

Hence show that the luminosity depends on the core mass as

$$L \propto M_1^{32/3}.$$

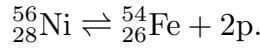
What happens to L if mass is removed from the stellar envelope?

3. At very high temperatures nuclear burning leads to an equilibrium distribution of isotopes around the iron group. In this nuclear statistical equilibrium, the abundance by number of an isotope ${}^A_Z S$ is,

$$N(A, Z) \propto n_p^Z n_n^{A-Z} T^{\frac{3}{2}(1-A)} \exp\left(\frac{Q(A, Z)}{kT}\right),$$

where $Q(A, Z)$ is the binding energy of the isotope, n_p and n_n are the number density of free protons and neutrons respectively.

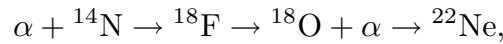
Derive an expression for the ratio of the abundances of ${}^{56}\text{Ni}$ and ${}^{54}\text{Fe}$ via the reversible reaction,



As the reactions proceed the ratio of the mean number of protons to the mean number of neutrons, $\frac{\langle Z \rangle}{\langle N \rangle}$, is preserved. When $\frac{\langle Z \rangle}{\langle N \rangle} = 1$ if all material is converted to either ${}^{56}\text{Ni}$, ${}^{54}\text{Fe}$ or n_{p} what is n_{p} ? Estimate the temperature at which $\frac{N({}^{56}\text{Ni})}{(N({}^{54}\text{Fe}))^3}$ is a maximum.

Outline when this equation can be used to calculate the outcome of nuclear reactions in the life of a massive star. What are the factors that cause $\frac{\langle Z \rangle}{\langle N \rangle}$ to decrease and when are they important to consider?

4. In most nuclear reactions throughout a star's lifetime $\frac{\langle Z \rangle}{\langle N \rangle} = 1$. However during helium burning nitrogen reacts as follows,



which forms nuclei with a ratio of $\frac{\langle Z \rangle}{\langle N \rangle} = \frac{5}{6}$. Nitrogen becomes the most abundant element after hydrogen burning via the CNO cycle. Assume it contributes the entire metallicity of a star at that point. If ${}^{22}\text{Ne}$ is the only element that contributes excess neutrons in a carbon-oxygen white dwarf and everything other than ${}^{22}\text{Ne}$ is converted to ${}^{12}\text{C}$ and ${}^{16}\text{O}$ in equal amounts by mass. Show that,

$$\frac{\langle Z \rangle}{\langle N \rangle} = \frac{11 - \mathbf{Z}}{11 + \mathbf{Z}}$$

where \mathbf{Z} is the metallicity mass fraction.

A type Ia supernova progenitor is a carbon-oxygen white dwarf. Describe the light curve of a type Ia supernova and how it is powered.

If a zero metallicity supernova forms $1M_{\odot}$ of nickel, calculate to 2 significant figures how much nickel a solar metallicity ($\mathbf{Z} = \frac{1}{50}$) type Ia supernova produces assuming the equilibrium is reached at the same temperature and density and that the only products are ${}^{56}\text{Ni}$ and ${}^{54}\text{Fe}$.

When the luminosity of a type Ia supernova is assumed to be a standard candle and calibrated at solar metallicity, show that the error in the distance to an extremely low metallicity type Ia supernova is about 3%.

[You may find it useful to know that $(Q(56, 28) - Q(54, 26))/k = -2.7 \times 10^{10}\text{K}$ and $\sqrt{550/523} = 1.03$]

5. A white dwarf has a helium core of mass M and radius R and a thin hydrogen-rich, non-degenerate envelope of mass $M_{\text{env}} \ll M$ and thickness $H \ll R$. Writing $z = r - R$, the surface density $\Sigma(z)$ in the envelope is defined by $d\Sigma = -\rho dz$ with $\Sigma = 0$ at $z = H$ and $\Sigma = \Sigma_o = M_{\text{env}}/4\pi R^2$ at $z = 0$. Show that, to a good approximation, in the envelope $P = \Sigma g$ where $g = GM/R^2$.

Show that the approximate equations for the structure of the envelope are

$$\frac{dF}{d\Sigma} = -\epsilon \quad \text{and} \quad \frac{dT}{d\Sigma} = \frac{3\kappa F}{4acT^3},$$

where $F = L_r/4\pi R^2$.

The opacity is of the form $\kappa = \kappa_0\rho T^{-3}$ where κ_0 is constant. The white dwarf's luminosity is produced solely by hydrogen burning in the envelope. The hydrogen burns steadily, mainly at the base of the envelope with $\epsilon = \epsilon_0\rho T^{25}$, where ϵ_0 is a constant. Radiation pressure can be neglected. Setting $y = T^8$ and $x = \frac{1}{2}\Sigma^2$ show that the structure equations can be written in the form

$$\frac{d^2y}{dx^2} = -\omega^2 y^3$$

where ω^2 is a positive constant.

Show that the appropriate boundary conditions are: at $x = 0$, $y = 0$ and $\frac{dy}{dx} = \frac{AL}{4\pi R^2}$, where A is a constant; at $x = \frac{1}{2}\Sigma_0^2$, $y = T_0^8$ and $\frac{dy}{dx} = 0$.

Multiply both sides by $\frac{dy}{dx}$, integrate and use the boundary conditions to show that

$$T_0 \propto \left(\frac{L}{4\pi R^2}\right)^{1/16}$$

Then integrate again and show that

$$\frac{L}{4\pi R^2} \propto \left(\frac{M_{\text{env}}}{4\pi R^2}\right)^{-4}$$

Comment on the stability of hydrogen burning in a white dwarf envelope and what might happen as a white dwarf accretes hydrogen material from a companion donor star.