Part III Magnetohydrodynamics
Michaelmas 2015

Examples III

1. Consider axisymmetric convection in a vertical cylinder of radius a and height h, with zero-stress boundary conditions for the velocity, and vertical magnetic field, on the curved sidewalls, and the same boundary conditions as for the plane layer problem at the top and bottom boundaries. Write down the equation satisfied by the flux function $\chi(r, z)$ in the steady state. Now concentrate on the flow and field near the axis where $\mathbf{u} = (-\frac{1}{2}r df(z)/dz, f(z))$. Seek solutions in the form of a thin flux tube near the axis when R_m is large. The total flux in the tube is Φ. Ignoring the z-derivatives in the diffusive term in the induction equation compared with the radial derivatives, find a solution for the flux function χ in the form of a Gaussian: $\chi = h(z)\exp(-r^2 g(z))$. Also find the Lorentz force due to this flux tube.

   For a convective cell of height d, $f(z)$ can be modelled by $U\sin(\pi z/d)$. For what range of z does the solution found above give an accurate representation of the full solution when $\mathbf{B}$ is constrained to be vertical at $z = 0, d$?

2. A simple model of the α-effect with broken symmetry is given by the equation for $\mathbf{B} = (B_x(z), B_y(z), 0)e^{\sigma t}$

$$\sigma \mathbf{B} = \alpha \hat{\mathbf{z}} \times \frac{\partial \mathbf{B}}{\partial z} - \Omega \frac{\partial \mathbf{B}}{\partial z} + \eta \frac{\partial^2 \mathbf{B}}{\partial z^2},$$

   where α, Ω are constants. Using the ansatz $\mathbf{Q} = \mathbf{B} + i\hat{\mathbf{z}} \times \mathbf{B}$, or otherwise, show that

   (i) for periodic boundary conditions with period $2\pi/k$ there are growing solutions for $\alpha^2 > (\eta/k)^2$.

   (ii) for boundary conditions $\mathbf{B} = 0$, $z = 0, L$ $(L \gg 1)$ there is growth only if $\alpha^2 > \Omega^2$. Why does this result not contradict that of part (i)? Sketch the amplitude of $\mathbf{B}$ as a function of z when $\Omega > 0$.

3. In the earth's core, a plausible balance of forces is between Lorentz and Coriolis forces, buoyancy forces and the pressure gradient, inertia and viscosity being neglected. This leads to the so called magnetostrophic equation:

$$2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla P + Q\hat{\mathbf{r}} + \mathbf{B} \cdot \nabla \mathbf{B}, \quad \nabla \cdot \mathbf{u} = 0,$$

   where Q is the buoyancy force, which is to be solved for $\mathbf{u}$ in a spherical shell of inner radius b and outer radius a, subject to the boundary condition $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$ at $r = a, b$.

   Show that for solutions to exist it is necessary that *Taylor's condition*, namely

$$T(s) = \int_{z_b(s)}^{z_t(s)} \int_0^{2\pi} \mathbf{B}(s, z, \phi) \cdot \nabla B_\phi(s, z, \phi)\, dz\, d\phi = 0,$$

   holds for every s where the integrals are taken along cylinders of constant s. Explain the difference in the geometry in the two cases $s < b$ and $b < s < a$, and show that in the former case there are in fact two integrals to be satisfied.

Assuming that $T(s) \equiv 0$ show that the solution for $\mathbf{u}$ is determined up to a geostrophic flow $V(s)\hat{\boldsymbol{\phi}}$.

Finally, assume that $\mathbf{B}, Q, \mathbf{u}$ are all axisymmetric, and that the regions $r < b$, $r > a$ are insulating. Show that the condition for solutions to exist can be reduced to the simplified form

$$\int_{z_b(s)}^{z_t(s)} B(s,z)\frac{\partial A(s,z)}{\partial z}\, dz = 0,$$

where $\mathbf{B} = B\hat{\boldsymbol{\phi}} + \nabla \times A\hat{\boldsymbol{\phi}}$.

4. Find the α-effect under the first-order smoothing approximation when the velocity $\mathbf{u}$ and fluctuating magnetic field $\mathbf{B}'$ is *non-helical*, isotropic forcing $\mathbf{f}$, and obey the equations

$$\frac{\partial \mathbf{u}}{\partial t} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla p + \overline{\mathbf{B}} \cdot \nabla \mathbf{B}' + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

$$\frac{\partial \mathbf{B}'}{\partial t} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B}'$$

.

5. The Parker interface dynamo. A simple model of the dynamo process near the base of the convection zone comprises two layers, where the mean magnetic field obeys the following equations, with $A, B \propto e^{ikx}$, k a horizontal wavenumber:

(a) $z > 0$; $\dot{A} = \alpha B + \eta_1(A_{zz} - k^2 A$, $\dot{B} = \eta_1(B_{zz} - k^2 B)$,

(b) $z < 0$; $\dot{A} = \eta_2(A_{zz} - k^2 A)$, $\dot{B} = \mathrm{i}k\Omega A + \eta_2(B_{zz} - k^2 B)$,

where α, Ω are constants (this is like an $\alpha\Omega$ dynamo with only the α-effect in $z > 0$ and only differential rotation in $z < 0$).

It is assumed that dynamo action is confined to the vicinity of $z = 0$ so that $A, B \to 0$ as $z \to \pm\infty$.

By applying appropriate boundary conditions at the interface, derive an equation for the growth rate σ of the dynamo implicitly in terms of $\alpha, \Omega, \eta_1, \eta_2, k$ and solve it in the special case $\eta_1 = \eta_2$.