Part III Stellar and Planetary Magnetic Fields Lent 2011

Examples I

1. A uniform magnetic field $\mathbf{B}_0 = (0, 0, B_0)$ in cartesian coordinates (x, y, z) is perturbed by the velocity field $\mathbf{u} = (U_0 \mathrm{sech}^2 kz, 0, 0)$. Find

   (i) the form of the initial perturbation to $\mathbf{B}$, neglecting diffusion

   (ii) the final steady state.

   What is the characteristic time for diffusion to become important?

2. Now repeat the calculation (i) above when $\mathbf{B}_0$ is replaced by $\mathbf{B}_1 = (0, 0, B_1 \cos \ell x)$, and $R_m = U_0/k\eta$ is large. By estimating the relative size of the diffusion term to the other terms in the equation, when the diffusionless solution is used, give an estimate of the time that elapses before diffusion becomes important in this case?

3. Consider the action of an axisymmetric stagnation point flow $\mathbf{u} = (-Ur/2a, 0, Uz/a$ (in cylindrical polar coordinates (r, ϕ, z)) on a magnetic field $\mathbf{B}$ that only has a z-component $B(r,t)$. Write down the equation satisfied by B, verify that no other components of B are induced by the flow, and show that there is a solution of the form

   $$B(r,t) = g(t)\exp(-f(t)s^2)),$$

   and find the equations satisfied by f and g. Verify that the total magnetic flux is conserved, and determine the final steady state of the system. Comment on the fact that the radial scale of this state is proportional to $R_m^{-1/2}$, where $R_m = Ua/\eta$.

4. Consider the action of a stagnation point flow $\mathbf{u} = A\mathbf{x}$, where the trace of the matrix A is zero.

   Show that the induction equation has a solution of the form $\mathbf{B} = \hat{\mathbf{B}}(t)e^{i\mathbf{k}(t)\cdot\mathbf{x}}$. Write down the equations for $\mathbf{B}$, $\mathbf{k}$,and use them to describe the subsequent evolution of $\mathbf{B}$.

5. Alfvén waves. Consider perturbations $\mathbf{u}(\mathbf{x}, t)$, $\mathbf{b}(\mathbf{x}, t)$ to a state with no velocity and uniform magnetic field $\mathbf{B}$. For an incompressible fluid the equations of motion and induction are

   $$(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + (\rho\mu_0)^{-1}\nabla \times \mathbf{b} \times (\mathbf{B} + \mathbf{b}) + \nu\nabla^2\mathbf{u}$$
   $$(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla)\mathbf{b} = (\mathbf{B} + \mathbf{b})\cdot\nabla\mathbf{b} + \eta\nabla^2\mathbf{b}$$
   $$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{b} = 0$$

   (i) Linearize the equations for small $\mathbf{u}, \mathbf{b}$ and seek spatially periodic solutions of the form $\mathbf{u}$, $\mathbf{b} \propto \exp(\sigma t + i\mathbf{k}\cdot\mathbf{x})$. Find the decay rate σ of such disturbances, and show that when the *Lundqvist number* $\mathcal{L} = |\mathbf{B}|/(\sqrt{\mu_0\rho\nu\eta}|\mathbf{k}|)$ is sufficiently large then σ is complex, corresponding to (damped) waves. Find approximations to the roots when the *Magnetic Prandtl number* $P_m = \nu/\eta$ is very large or very small.

   (ii) Now ignore the diffusion terms, but retain the nonlinear terms. Show that the equations have exact solutions in the form of waves that travel at the *Alfvén velocity* $\mathbf{V} = \mathbf{B}/\sqrt{\mu_0\rho}$.

6. A solid circular cylinder of radius a rotates about the z-axis with angular velocity Ω. Far from the cylinder there is a uniform magnetic field B in the x direction. Outside the cylinder the material is insulating, while the cylinder itself is conducting with magnetic diffusivity η. The magnetic Reynolds number $R_m = \Omega a^2/\eta$ is very large.

   Assuming that the resulting steady magnetic field is two-dimensional (no z component, independent of z), and that $\mathbf{B} = \nabla \times (0, 0, A)$ in cylindrical polar coordinates (r, ϕ, z), show that $A = \mathbf{Re}\hat{A}(r)e^{i\phi}$ and that

$$i\Omega\hat{A} = \eta\left(\hat{A}_{rr} + \frac{1}{r}\hat{A}_r - \frac{1}{r^2}\hat{A}\right), \ r < a; \quad \hat{A}_{rr}+\frac{1}{r}\hat{A}_r-\frac{1}{r^2}\hat{A} = 0, \ r > a; \quad \hat{A} \sim Br, \ r \to \infty.$$

   Give the boundary conditions satisfied by $\hat{A}$ at $r = a$. Show that when R_m is large the magnetic field in the cylinder is confined to a thin boundary layer near $r = a$, of thickness $aR_m^{-\frac{1}{2}}$.

7. Consider the simple "One-dimensional" mean-field dynamo equations for a steady α^2-dynamo:

$$0 = \alpha_0 f(z)B + \eta\frac{d^2A}{dz^2}, \quad 0 = -\alpha_0\left(f(z)\frac{dA}{dz}\right)_z + \eta\frac{d^2B}{dz^2},$$

   where $B = A = 0$ at $z = \pm 1$. Suppose that $f(z)$ is an odd function of z. Then we can find dipole modes with A even and B odd in z, and quadrupole modes with B even, A odd. Solve the equations and show that the critical values of α_0 are the same for each parity.