Part III Stellar and Planetary Magnetic Fields
Lent 2011

Examples II

1. Turbulent diffusivity. Solve the steady induction equation for $\mathbf{B}'$ in the first order smoothing approximation:

$$0 = -\mathbf{u} \cdot \nabla \mathbf{B}_0 + \mathbf{B}_0 \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B}',$$

where $B_{0i} = Q_{ij} x_j$, Q_{ij} constant, and $\mathbf{u} = \int \hat{\mathbf{u}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}$. Find the expression for $\mathcal{E}$ in terms of Q_{ij} and show that when the statistics of $\mathbf{u}$ are isotropic $\nabla \times \mathcal{E}$ has the form of an additional diffusion term for the mean field $\mathbf{B}_0$.

2. Consider the mean emf due to a steady solenoidal velocity field $\mathbf{u}(\mathbf{x})$ at small magnetic Reynolds number R_m. The velocity field is monochromatic, so that $\nabla^2 \mathbf{u} = -\mathbf{u}$. Consider the induction equation in the form

$$0 = \mathbf{B} \cdot \nabla \mathbf{u} + \nabla \times (\mathbf{u} \times \mathbf{b}) + \frac{1}{R_m} \nabla^2 \mathbf{b},$$

where $\mathbf{B}$ is a constant vector, and $\mathbf{b}$ is the induced field. Show that the mean emf $\mathcal{E} = \overline{\mathbf{u} \times \mathbf{b}}$, where the overbar denotes an average over all space, can be written in the form

$$\mathcal{E} = R_m \mathcal{E}^{(1)} + R_m^2 \mathcal{E}^{(2)} + \ldots = R_m \overline{\mathbf{u} \times \mathbf{B} \cdot \nabla \mathbf{u}} + R_m^2 \overline{\mathbf{u} \times \nabla \times (\mathbf{u} \times \mathbf{B} \cdot \nabla \mathbf{u})} + \ldots$$

Show, without using Fourier decomposition, that if $\mathcal{E}_i^{(p)}$ can be written $\alpha_{ij}^{(p)} B_j$, then $\alpha^{(1)}$ is symmetric.

Derive the result

$$\mathcal{E}_i^{(2)} = -\overline{\frac{\partial u_j}{\partial x_i} (\mathbf{u} \times \mathbf{B} \cdot \nabla \mathbf{u})_j} \; .$$

Now suppose that $\mathbf{B} = (B, 0, 0)$, and that $\mathbf{u}$ depends only on x and y. Show that $\mathcal{E}_z^{(2)} = 0$, and that $\alpha_{12}^{(2)} \neq \alpha_{21}^{(2)}$ only if $\mathbf{u}|$ depends upon both x and y, and all three components of $\mathbf{u}$ are non-zero.

3. By analogy with lectures, find the α-effect under the first-order smoothing approximation when the velocity is driven by *non-helical* forcing, and obeys the equation

$$\frac{\partial \mathbf{u}}{\partial t} + 2\mathbf{\Omega} \times \mathbf{u} = -\nabla p + \overline{\mathbf{B}} \cdot \nabla \mathbf{B}' + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

4. Prove the result given in lectures for the α-effect, namely

$$\mathcal{E} \cdot \overline{\mathbf{B}} = \alpha_{ij} \overline{\mathbf{B}}_i \overline{\mathbf{B}}_j = -\frac{1}{\eta} \overline{\mathbf{B}' \cdot \nabla \times \mathbf{B}'},$$

for a mean field dynamo in a statistically steady state, stating carefully any assumptions you make. Verify that in the First Order Smoothing limit the expression agress with the result derived in lectures. *[Hint: consider the equation for the vector potential].*

5. Find conditions for the existence of a steady α^2-dynamo with uniform α satisfying

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\alpha \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

in a conducting sphere of radius a surrounded by insulator. (Use the poloidal-toroidal decomposition as for the free decay modes in lectures.) Sketch the lines of poloidal field and contours of toroidsl field in the case of an axisymmetric solution (which corresponds to the lowest value of $|\alpha|$.) Why is a model with constant α unsatisfactory as a consequence of the effect of rotation in small-scale flows?

6. A nonlinear model of Parker dynamo waves, incorporating the effects of induced zonal flow, takes the form of four ODEs:

$$\dot{A} = DB - A, \quad \dot{B} = iA(1 + U_0) - B + iU_2 A^*,$$
$$\dot{U_0} = \frac{i}{2}(A^* B - AB^*) - \nu_0 U_0,$$
$$\dot{U_2} = iAB - \nu_2 U_2.$$

Here A,B represent poloidal and toroidal fields, respectively, proportional to e^{ikx}, while U_0 (real) represents the zonal flow perturbation independent of x, and, U_2 the zonal flow perturbation proportional to e^{2ikx}.

(i) Verify that the nonlinear terms make no contribution to the rate of change of total "energy" $|A|^2 + |B|^2 + U_0^2 + |U_2|^2$.

(ii) Show that travelling wave solutions, with $A, B \propto e^{i\Omega t}$, U_0 steady and $U_2 \propto e^{2i\Omega t}$, can be found, and write down equations giving (in parametric form) the relation between D, Ω and $|A|^2$. Find approximate formulae for D and Ω when $|A|^2$ is (a) very small and (b) very large.