Part III Stellar and Planetary Magnetic Fields
Lent 2011

Examples III

1. Consider the problem of the onset of convection in a fluid layer in the presence of a vertical magnetic field. Show that the condition for the onset of steady two-dimensional convection (i.e. no velocity in the y-direction, no dependence on y, no applied electric field in the y-direction) does not depend on the boundary conditions satisfied by the magnetic field at the top and bottom of the layer. Is the same true for 3D disturbances?

2. A vertical flux tube has a magnetic field of the form $\mathbf{B} = (0, B_\phi(r), B_z(r))$, with $B_\phi = B_z = 0$ at $r = a$ in cylindrical polar coordinates (r, ϕ, z). $B_\phi = 0$ also at $r = 0$. The gas pressure at $r = a$ is p_e. It may be assumed that the gas pressure increases outwards and that $p(0) = 0$.

   Write down an equation for the pressure $p(r)$, $r \leq a$ and show that $B_z^2(0) \geq 2\mu_0 p_e$, but that $\frac{2}{a^2}\int_0^a rB_z^2\, dr \leq 2\mu_0 p_e$

3. Consider axisymmetric convection in a vertical cylinder of radius a and height h, with zero-stress boundary conditions for the velocity, and vertical magnetic field, on the curved sidewalls, and the same boundary conditions as for the plane layer problem at the top and bottom boundaries. By writing down the equation satisfied by the magnetic flux function $\chi(r, z)$ where

$$B_z = \frac{1}{r}\frac{\partial \chi}{\partial r}, \quad B_r = -\frac{1}{r}\frac{\partial \chi}{\partial z},$$

   or otherwise, show that in the steady state $2\pi\int_0^a\int_0^h r dr\, dz B_Z(r, z) = \pi a^2 h B_0$, $\int_0^h B_z(a, z) dz = hB_0$, where B_0 is the magnitude of the vertical imposed field.

   Suppose now that essentially all the flux at any height is confined to thin regions near $r = 0, a$ of thickness $O(R_m^{-1/2})$. What can you say about the proportion of flux near the axis?

4. Consider the same steady state problem as above, and concentrate on the flow and field near the axis where $\mathbf{u} = (-\frac{1}{2}rdf(z)/dz, f(z))$. Seek solutions in the form of a thin flux tube near the axis when R_m is large. The total flux in the tube is Φ. Ignoring the z-derivatives in the diffusive term in the induction equation compared with the radial derivatives, find a solution for the flux function χ in the form of a Gaussian: $\chi = h(z)\exp(-r^2 g(z))$. Also find the Lorentz force due to this flux tube.

   For a convective cell of height d, $f(z)$ can be modelled by $U\sin(\pi z/d)$. For what range of z does the solution found above give an accurate representation of the full solution when $\mathbf{B}$ is constrained to be vertical at $z = 0, d$?

5. A perfect gas is in hydrostatic equilibrium with pressure $p(z)$, density $\rho(z)$ and temperature $T(z)$. A small horizontal flux tube with field strength B is introduced into the gas at $z = 0$, with a temperature T_B such that the density inside the tube is the same as the external density.

   (i) Determine T_B.

   (ii) The tube is displaced to a height dz. Assuming that normal stresses remain in balance, and that the displacement is adiabatic, find the density $\rho(0) + \delta\rho$ inside the tube, and give conditions on B_0 and the properties of the atmosphere such that the tube continues to rise.

6. The region $z < 0$ contains fluid of uniform density ρ_1, uniform horizontal magnetic field $\mathbf{B}_0$, and moves at horizontal velocity $(U, 0, 0)$. In $z > 0$ the density is ρ_2, the magnetic field is zero and the velocity is $(-U, 0, 0)$.

   Following the method in lectures for the case $U = 0$, $\mathbf{B}_0 = (0, B_0, 0)$, find the growth rate for disturbances with horizontal dependence of the form $\exp(i\mathbf{k}\cdot\mathbf{x})$, $\mathbf{k} = (l, m)$.

7. Consider the solution of the magnetostrophic equation in a spherical shell of inner and outer radii a, b respectively, with a force applied of the form $\hat{\mathbf{f}}(r, z)e^{im\phi}$ in cylindrical polar coordinates (r, ϕ, z), and m a non-zero integer. Show by construction that there is a solution for the velocity field with the same ϕ dependence, whatever the form of $\hat{\mathbf{f}}$, but that the solution is discontinuous in general across the tangent cylinder $r = a$. Give conditions on $\hat{\mathbf{f}}$ that ensure that the r-component of the velocity is continuous.