Example Sheet 1

1. *Revision of Keplerian orbits*

The equation of motion of a test particle in the gravitational field of a point mass M is

$$\ddot{\boldsymbol{r}} = -\frac{GM\boldsymbol{r}}{|\boldsymbol{r}|^3} .$$

Show that the motion is confined to a plane containing the central mass. Introduce polar coordinates (r, ϕ) in the plane and deduce that

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2} ,$$

$$r^2\dot{\phi} = h = \text{constant} .$$

Find an equation for the shape $r(\phi)$ of the orbit and show that the general solution is

$$r = \frac{\lambda}{1 + e\cos(\phi - \omega)} ,$$

where e and ω are arbitrary constants, and $\lambda = h^2/GM$. Sketch the orbit for the cases $e = 0$, $0 < e < 1$, $e = 1$ and $e > 1$.

Show that the orbit of least energy for a given angular momentum is a circular orbit.

2. *Precession in Newtonian dynamics*

(i) Given an axisymmetric gravitational potential $\Phi(r, z)$ for which $z = 0$ is a plane of symmetry, explain how to determine the angular velocity $\Omega(r)$ of circular orbits in the plane $z = 0$, as well as the epicyclic frequency $\kappa(r)$ and the vertical frequency $\Omega_z(r)$.

(ii) In regions of space where Φ satisfies Laplace's equation (i.e. away from the important sources of the gravitational field), show that the three frequencies are related by

$$\kappa^2 + \Omega_z^2 = 2\Omega^2 . \quad (1)$$

What does this relation imply for the directions of apsidal and nodal precession?

(iii) The exterior gravitational potential of a rotating planet or star can be written in the form

$$\Phi = -\frac{GM}{R}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_\mathrm{e}}{R}\right)^n P_n(\cos\theta)\right] ,$$

where M is the mass of the body, R_e is its equatorial radius, (R, θ, ϕ) are *spherical* polar coordinates, J_n is a dimensionless coefficient and P_n is the Legendre polynomial of degree n.

Explain why this form is reasonable, and why it is reasonable that $J_n = 0$ for odd values of n.

Find expressions for $\Omega(r)^2$, $\kappa(r)^2$ and $\Omega_z(r)^2$. Verify that equation (1) is satisfied. Show that the leading approximations for the apsidal and nodal precession rates far from the planet or star are

$$\Omega - \kappa \approx +\frac{3}{2} J_2 \left(\frac{R_e}{r}\right)^2 \Omega \,,$$

$$\Omega - \Omega_z \approx -\frac{3}{2} J_2 \left(\frac{R_e}{r}\right)^2 \Omega \,.$$

[*Hint*: $P_n(x)$ satisfies Legendre's equation $(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$. For positive integers m,

$$P_{2m}(0) = \frac{(-1)^m (2m)!}{2^{2m}(m!)^2} \,.$$

For Saturn, $J_2 \approx 1.63 \times 10^{-2}$ and $J_4 \approx -9.4 \times 10^{-4}$.]

3. *Motion in the Kerr metric*

The exterior of a black hole is described by the Kerr metric, which contains two parameters: M, the mass of the black hole, and a, the dimensionless spin parameter, which satisfies $-1 < a < 1$. A relativistic treatment of test-particle orbits in the equatorial plane leads to the expressions

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2} \frac{1}{1 + a x^{-3/2}} \,,$$

$$\kappa^2 = \Omega^2 \left(1 - 6x^{-1} + 8a x^{-3/2} - 3a^2 x^{-2}\right) \,,$$

$$\Omega_z^2 = \Omega^2 \left(1 - 4a x^{-3/2} + 3a^2 x^{-2}\right) \,,$$

where $x = r/(GM/c^2)$ is the orbital radius in gravitational units. (Here r is the Boyer–Lindquist radial coordinate, and the frequencies are those measured by an observer at infinity. The event horizon is located at $x = 1 + (1-a^2)^{1/2}$. Since $\Omega > 0$, choosing $a > 0$ corresponds to a prograde orbit in the same direction as the spin of the black hole, while $a < 0$ corresponds to a retrograde orbit.)

(i) Assuming the above expressions, deduce that circular orbits outside the event horizon, but sufficiently close to it, are unstable.

(ii) Show that the leading approximations for the apsidal and nodal precession rates far from the black hole are

$$\Omega - \kappa \approx 3x^{-1}\Omega\,,$$

$$\Omega - \Omega_z \approx 2ax^{-3/2}\Omega\,.$$

4. **Steady state disk emission**

The diffusion equation governing the evolution of a Keplerian accretion disk around a compact object of mass M_* is

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{2\pi r}\partial_r \mathcal{F},$$

where the mass flux $\mathcal{F}$ is defined by

$$\mathcal{F} = 6\pi r^{1/2}\partial_r\left(r^{1/2}\overline{\nu}\Sigma\right),$$

with Σ the disk surface density and $\overline{\nu}$ the mean turbulent viscosity. The viscous torque vanishes at the radius of the central object $r = r_*$, and the disk is supplied with mass at its outer radius $r = r_{\text{out}} \gg r_*$ at a rate $\dot{M}$.

(i) Calculate the steady state disk structure $\Sigma = \Sigma(r)$.

(ii) Assume the disk is in steady state energy balance so that cooling balances heating: $\mathcal{C} = \mathcal{H}$. The disk surfaces radiate as blackbodies so that $\mathcal{C} = 2\sigma T^4$, where σ is the Stefan-Boltzmann constant and T is the effective temperature. The heating rate is given by $\mathcal{H} = \frac{9}{4}\overline{\nu}\Sigma\Omega^2$, where $\Omega = \Omega(r)$ is the orbital frequency. Show that

$$T = T_{\text{in}}\left(\frac{r_*}{r}\right)^{3/4}\left(1 - \sqrt{\frac{r_*}{r}}\right)^{1/4},$$

where $T_{\text{in}} = (3GM_*\dot{M}/8\pi\sigma r_*^3)^{1/4}$.

(iii) Consider a white dwarf and a neutron star of roughly the same mass and accreting at the same rate. The white dwarf has radius $\sim 10^4$ km, while the neutron star has a radius of ~ 10 km. How much hotter is the disk around the neutron star and how will this influence its emitted spectrum? Suppose both stars are accreting at a rate of $\dot{M} \sim 10^{16}$ g/s and have a mass of roughly a solar mass. Calculate the maximum effective temperature (in Kelvin) of the accretion disks encircling the two stars, and their expected frequencies of emission.

Consider a black hole accreting near the Eddington limit, so that $\dot{M} \sim 4\pi GM_*/(c\kappa_T)$, where κ_T is the Thomson opacity. Derive the scaling $T \sim M_*^{-1/4}$. Comment on the temperature difference between a disk around a supermassive black hole (a billion solar masses) versus a stellar mass black hole in an X-ray binary (ten solar masses).

(iv) The spectral energy flux of the disk is given by

$$F_\nu \propto \nu^3 \int_{r_*}^{r_{\rm out}} \frac{r}{e^{h\nu/kT}-1}\, dr,$$

where ν is the frequency of the radiation, and h and k are constants.

Show that $F_\nu \propto \nu^2$ in the low frequency limit $\nu \ll kT_{\rm out}/h$, where $T_{\rm out}$ is the temperature at the outer boundary of the disk. Which region of the disk is primarily emitting in this range?

Next show that $F_\nu \propto \nu^{1/3}$ for intermediate frequencies $kT_{\rm out}/h \ll \nu \ll kT_{\rm in}/h$. You may assume for this calculation that $T \approx T_{\rm in}(r_*/r)^{3/4}$. In both parts of the question the constant of proportionality involves a dimensionless integral you need not evaluate.

(v) The luminosity of an annulus of disk between $r = r_1$ and $r = r_2$ is

$$L = \int_{r_1}^{r_2} 2\pi r \mathcal{H}\, dr.$$

Show that the total luminosity of the disk is approximately $GM_*\dot{M}/(2r_*)$. Next show that this is only half the potential energy liberated by accretion of material to the radius $r = r_*$. What happens to the remaining energy? It's dissipated in the star-disk boundary layer!

(vi) The disk connects to the central star via a boundary layer of thickness $\sim H$, the disk's local scale height. Assuming the boundary layer emits as a blackbody, show that the effective temperature of the boundary layer is

$$T_{BL} \sim \left(\frac{r_*}{H}\right)^{1/4} T_{\rm in}\,.$$

What can you say about the boundary layer's emission in comparison to the disk's?

5. *Linear diffusion equation*

The surface density $\Sigma(r,t)$ of a Keplerian accretion disc evolves according to the diffusion equation

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right)\right], \qquad (2)$$

where $\bar{\nu}$ is the mean effective kinematic viscosity.

Show that, for a particular choice of the function $\bar{\nu} = \bar{\nu}(r)$, equation (2) can be reduced to the classical diffusion equation

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial y^2}, \qquad (3)$$

by a suitable transformation of variables.

Formulate the conservation laws of mass and angular momentum in terms of the variables g and y. What are the appropriate solutions of equation (3) representing the spreading of an initially very narrow ring subject to the boundary conditions that

(i) mass can be accreted at $r = 0$, but no torque is exerted there?

(ii) no mass is accreted at $r = 0$, but a torque is exerted there?

6. *Nonlinear diffusion equation*

(i) Consider a disc with the viscosity law

$$\bar{\nu} = Ar^2\Sigma\,,$$

where A is a constant. Show that solutions of equation (2) of the form

$$\Sigma = \sigma(t)\left\{\left[\frac{R(t)}{r}\right]^a - 1\right\}, \qquad r \leqslant R(t)\,,$$

exist for only two non-zero values of the parameter a, namely $a = 1$ and $a = 5/4$. In each case, solve for $\sigma(t)$ and $R(t)$, assuming that $R(0) = 0$.

(ii) For the solution with $a = 1$, show that the mean radial velocity in the disc is

$$\bar{u}_r = -\frac{(R - 5r)}{10t}\,.$$

Determine the trajectories of fluid elements moving with the mean radial velocity, and deduce that almost every fluid element is accreted in a finite time. For the solution with $a = 5/4$, show that $\bar{u}_r$ is strictly positive.

(iii) Investigate whether mass and angular momentum are globally conserved in either of the two solutions. Comment on the likely significance of these special solutions among solutions of the nonlinear diffusion equation as an initial-value problem with various initial and boundary conditions.

7. *Vertical structure with radiation pressure*

The vertical structure of a thin, Keplerian accretion disc with constant (Thomson) opacity and a mixture of gas and radiation is governed by the equations

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z\,,$$

$$\frac{\partial F}{\partial z} = \mu\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2,$$

$$F = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z}\,,$$

$$p = \frac{k\rho T}{\mu_\mathrm{m} m_\mathrm{p}} + \frac{4\sigma T^4}{3c}\,.$$

(i) Show that there is only one possible value of the effective viscosity μ of an accretion disc in which the gas pressure is negligible compared to the radiation pressure. How does this value compare with the viscosity of water?

(ii) According to one theory, whatever the ratio of gas and radiation pressures, the effective viscosity is always related to the gas pressure p_{g} by $\mu = \alpha p_{\mathrm{g}}/\Omega$, where α is a constant. Show that the temperature then satisfies an equation of the form

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial T^4}{\partial z}\right) + A\Omega T = 0\,,$$

where A is a constant to be determined.

Adopt a mass-weighted vertical coordinate ζ, defined via $d\zeta \propto \rho dz$ with $\zeta = 0$ at the midplane and $\zeta = 1$ at the upper surface. Hence find an explicit expression for the mean kinematic viscosity $\bar{\nu}(r, \Sigma)$ of the disc. You may assume that the 'zero boundary conditions' apply, and that there exists a unique non-trivial solution $t(\zeta)$ of the dimensionless boundary-value problem

$$\frac{\mathrm{d}^2 t^4}{\mathrm{d}\zeta^2} + t = 0, \qquad t'(0) = t(1) = 0\,.$$

Your answer may involve an integral of $t(\zeta)$, which need not be evaluated.

8. **Steady alpha disc**

Write down the relation that must hold between $\bar{\nu}$ and Σ in a steady, Keplerian accretion disc with inner radius r_{in} (at which no torque is exerted) and accretion rate $\dot{M}$. Combine this with an order-of-magnitude treatment of the vertical structure of a gas pressure-dominated disc with Thomson opacity to deduce that

$$H \sim \alpha^{-1/10}\Omega^{-7/10}(f\dot{M})^{1/5}\left(\frac{\mu_{\mathrm{m}} m_{\mathrm{p}}}{k}\right)^{-2/5}\left(\frac{\sigma}{\kappa}\right)^{-1/10}\,,$$

where $f = 1 - (r_{\mathrm{in}}/r)^{1/2}$. (Notice how insensitive this result is to the poorly known value of α.) Hence show that the disc is very slightly flared, in the sense that H/r increases very slowly with increasing r.

Repeat the calculation for Kramers opacity, $\kappa = C_\kappa \rho T^{-7/2}$, to obtain

$$H \sim \alpha^{-1/10}\Omega^{-3/4}(f\dot{M})^{3/20}\left(\frac{\mu_{\mathrm{m}} m_{\mathrm{p}}}{k}\right)^{-3/8}\left(\frac{\sigma}{C_\kappa}\right)^{-1/20}\,.$$

(These results are approximately valid in the inner and outer parts, respectively, of accretion discs in interacting binary stars.)

Please send any comments and corrections to hl278@cam.ac.uk
Answers to starred questions may be submitted for marking and feedback.