Example Sheet 1

1. *Revision of Keplerian orbits*

The equation of motion of a test particle in the gravitational field of a point mass M is

$$\ddot{\mathbf{r}} = -\frac{GM\mathbf{r}}{|\mathbf{r}|^3}\,.$$

Show that the motion is confined to a plane containing the central mass. Introduce polar coordinates (r, ϕ) in the plane and deduce that

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2}\,,$$

$$r^2\dot{\phi} = h = \text{constant}\,.$$

Find an equation for the shape $r(\phi)$ of the orbit and show that the general solution is

$$r = \frac{\lambda}{1 + e\cos(\phi - \omega)}\,,$$

where e and ω are arbitrary constants, and $\lambda = h^2/GM$. Sketch the orbit for the cases $e = 0$, $0 < e < 1$, $e = 1$ and $e > 1$.

Show that the orbit of least energy for a given angular momentum is a circular orbit.

2. *Precession in Newtonian dynamics*

(i) Given an axisymmetric gravitational potential $\Phi(r, z)$ with reflectional symmetry in the midplane $z = 0$, explain how to determine the angular velocity $\Omega(r)$ of circular orbits in the midplane, as well as the radial (epicyclic) frequency $\Omega_r(r)$ and the vertical frequency $\Omega_z(r)$.

(ii) In regions of space where Φ satisfies Laplace's equation (i.e. away from the important sources of the gravitational field), show that the three frequencies are related by

$$\Omega_r^2 + \Omega_z^2 = 2\Omega^2\,. \tag{1}$$

What does this relation imply for the directions of apsidal and nodal precession?

(iii) Consider a binary system consisting of two stars of total mass $M = M_1 + M_2$ in a circular orbit of separation a. First consider the case of a *circumstellar disc* (with outer radius $< a$) that surrounds star 1. Using cylindrical polar coordinates (r, ϕ, z)

in a non-rotating frame centred on star 1, show that when the gravitational potential of the binary is averaged, either in time or in azimuth, it has the expansion

$$-\frac{GM_1}{\sqrt{r^2 + z^2}} - \frac{GM_2}{a}\left[1 + \frac{(r^2 - 2z^2)}{4a^2} + O\left(\frac{|\mathbf{r}|^4}{a^4}\right)\right].$$

Evaluate Ω^2, Ω_r^2 and Ω_z^2 to the same order of accuracy for orbits in this potential, and verify that equation (1) holds. Deduce that the precession frequencies are

$$\Omega - \Omega_r \approx +\frac{3}{4}\frac{M_2}{M_1}\frac{r^3}{a^3}\sqrt{\frac{GM_1}{r^3}}, \qquad \Omega - \Omega_z \approx -\frac{3}{4}\frac{M_2}{M_1}\frac{r^3}{a^3}\sqrt{\frac{GM_1}{r^3}}.$$

(iv) Now consider the case of a *circumbinary disc* (with inner radius $> a$) that surrounds the binary system. Using cylindrical polar coordinates (r, ϕ, z) in a non-rotating frame centred on the centre of mass of the binary, show that the averaged gravitational potential of the binary has the expansion

$$-\frac{GM}{\sqrt{r^2 + z^2}}\left[1 + \frac{M_1 M_2}{M^2}\frac{(r^2 - 2z^2)a^2}{4(r^2 + z^2)^2} + O\left(\frac{a^4}{|\mathbf{r}|^4}\right)\right].$$

Evaluate Ω^2, Ω_r^2 and Ω_z^2 to the same order of accuracy for orbits in this potential, and verify that equation (1) holds. Deduce that the precession frequencies are

$$\Omega - \Omega_r \approx +\frac{3}{4}\frac{M_1 M_2}{M^2}\frac{a^2}{r^2}\sqrt{\frac{GM}{r^3}}, \qquad \Omega - \Omega_z \approx -\frac{3}{4}\frac{M_1 M_2}{M^2}\frac{a^2}{r^2}\sqrt{\frac{GM}{r^3}}.$$

3. *Motion in the Kerr metric*

The exterior of a black hole is described by the Kerr metric, which contains two parameters: M, the mass of the black hole, and a, the dimensionless spin parameter, which satisfies $-1 < a < 1$. A relativistic treatment of test-particle orbits in the equatorial plane leads to the expressions

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2}\frac{1}{1 + ax^{-3/2}},$$

$$\Omega_r^2 = \Omega^2\left(1 - 6x^{-1} + 8ax^{-3/2} - 3a^2x^{-2}\right),$$

$$\Omega_z^2 = \Omega^2\left(1 - 4ax^{-3/2} + 3a^2x^{-2}\right),$$

where $x = r/(GM/c^2)$ is the orbital radius in gravitational units. [Here r is the Boyer–Lindquist radial coordinate, and the frequencies are those measured by an observer at infinity. The event horizon is located at $x = 1 + \sqrt{1 - a^2}$. Since $\Omega > 0$, choosing $a > 0$ corresponds to a prograde orbit in the same direction as the spin of the black hole, while $a < 0$ corresponds to a retrograde orbit.]

(i) Assuming the above expressions, deduce that circular orbits outside the event horizon, but sufficiently close to it, are unstable.

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(ii) Show that the leading approximations for the apsidal and nodal precession rates far from the black hole are

$$\Omega - \Omega_r \approx 3x^{-1}\Omega\,, \qquad \Omega - \Omega_z \approx 2ax^{-3/2}\Omega\,.$$

4. *Linear diffusion equation*

The surface density $\Sigma(r,t)$ of a Keplerian accretion disc evolves according to the diffusion equation

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right)\right]\,, \tag{2}$$

where $\bar{\nu}$ is the mean effective kinematic viscosity.

Show that, for a particular choice of the function $\bar{\nu} = \bar{\nu}(r)$, equation (2) can be reduced to the classical diffusion equation

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial y^2}\,, \tag{3}$$

by a suitable transformation of variables.

Formulate the conservation laws of mass and angular momentum in terms of the variables g and y. What are the appropriate solutions of equation (3) representing the spreading of an initially very narrow ring subject to the boundary conditions that

(i) mass can be accreted at $r = 0$, but no torque is exerted there?

(ii) no mass is accreted at $r = 0$, but a torque is exerted there?

5. *Nonlinear diffusion equation*

(i) Consider a disc with the viscosity law

$$\bar{\nu} = Ar^2\Sigma\,,$$

where A is a constant. Show that solutions of equation (2) of the form

$$\Sigma = \sigma(t)\left\{\left[\frac{R(t)}{r}\right]^a - 1\right\}\,, \qquad r \leqslant R(t)\,,$$

exist for only two non-zero values of the parameter a, namely $a = 1$ and $a = 5/4$. In each case, solve for $\sigma(t)$ and $R(t)$, assuming that $R(0) = 0$.

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(ii) For the solution with $a = 1$, show that the mean radial velocity in the disc is

$$\bar{u}_r = -\frac{(R - 5r)}{10t}.$$

Determine the trajectories of fluid elements moving with the mean radial velocity, and deduce that almost every fluid element is accreted in a finite time. For the solution with $a = 5/4$, show that $\bar{u}_r$ is strictly positive.

(iii) Investigate whether mass and angular momentum are globally conserved in either of the two solutions. Comment on the likely significance of these special solutions among solutions of the nonlinear diffusion equation as an initial-value problem with various initial and boundary conditions.

6. *Spreading of a narrow ring*

In the local approximation, with $r = r_0 + x$ and $|x| \ll r_0$, equation (2) reduces to

$$\frac{\partial \Sigma}{\partial t} = 3\frac{\partial^2}{\partial x^2}(\bar{\nu}\Sigma).$$

If $\bar{\nu} \propto \Sigma^p$, with $p > 0$, show that algebraic solutions exist that are of the form

$$\Sigma = \sigma(t)\left\{1 - \left[\frac{x}{w(t)}\right]^a\right\}^b, \qquad |x| \leqslant w(t),$$

for suitable choices of a and b. Solve for $\sigma(t)$ and $w(t)$. Comment on how mass and angular momentum are conserved in these solutions in the local approximation, and on how the total orbital energy evolves. Discuss the limit $p \to 0$.

7. *Vertical structure with radiation pressure*

The vertical structure of a thin, Keplerian accretion disc with constant (Thomson) opacity and a mixture of gas and radiation is governed by the equations

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z,$$

$$\frac{\partial F}{\partial z} = \rho\nu\left(r\frac{d\Omega}{dr}\right)^2,$$

$$F = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z},$$

$$p = \frac{\mathscr{R}\rho T}{\mu} + \frac{4\sigma T^4}{3c}.$$

(i) Show that there is only one possible value of the effective viscosity $\rho\nu$ of an accretion disc in which the gas pressure is negligible compared to the radiation pressure. How does this value compare with the viscosity of water?

(ii) According to one theory, whatever the ratio of gas and radiation pressures, the effective viscosity is always related to the gas pressure p_g by $\rho\nu = \alpha p_\mathrm{g}/\Omega$, where α is a constant. Show that the temperature then satisfies an equation of the form

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial T^4}{\partial z}\right) + A\Omega T = 0\,,$$

where A is a constant to be determined.

Adopt a mass-weighted vertical coordinate ζ, defined via $d\zeta \propto \rho\,dz$ with $\zeta = 0$ at the midplane and $\zeta = 1$ at the upper surface. Hence find an explicit expression for the mean kinematic viscosity $\bar{\nu}(r,\Sigma)$ of the disc. You may assume that the 'zero boundary conditions' apply, and that there exists a unique non-trivial solution $t(\zeta)$ of the dimensionless boundary-value problem

$$\frac{d^2 t^4}{d\zeta^2} + t = 0, \qquad t'(0) = t(1) = 0\,.$$

Your answer may involve an integral of $t(\zeta)$, which need not be evaluated.

8. *Steady alpha disc*

Write down the relation that must hold between $\bar{\nu}$ and Σ in a steady, Keplerian accretion disc with inner radius r_in (at which no torque is exerted) and accretion rate $\dot{M}$. Combine this with a treatment of the vertical structure of a gas pressure-dominated disc with a constant viscosity parameter α and a power-law opacity $\kappa = C_\kappa \rho^x T^y$ (as in lectures) to deduce that

$$H^{3x-2y+10} = \frac{27}{64\lambda}\alpha^{-(x+1)}\frac{C_\kappa}{\sigma}\left(\frac{\mathscr{R}}{\mu}\right)^{4-y}\left(\frac{f\dot{M}}{3\pi}\right)^{x+2}\Omega^{-(x-2y+7)},$$

where $f = 1 - (r_\mathrm{in}/r)^{1/2}$ and λ is the dimensionless cooling rate as defined in lectures.

The Thomson ($x = y = 0$) and Kramers ($x = 1$, $y = -7/2$) opacity laws are approximately valid in the inner and outer parts, respectively, of accretion discs in interacting binary stars. Show that, in each case, the aspect ratio H/r is very weakly dependent on the values of α and C_κ, and that it increases very slowly with r, so that the disc is very slightly flared.

Estimate the value of H/r in the case of Thomson opacity ($C_\kappa \approx 0.33\,\mathrm{cm^2\,g^{-1}}$) for a white dwarf ($M \approx 1\,M_\odot$) accreting at $\dot{M} \approx 10^{-9}\,M_\odot\,\mathrm{yr^{-1}}$.

Please send any comments and corrections to `gio10@cam.ac.uk`
Answers to questions 2 and 5 may be submitted for marking.