Example Sheet 2

1. *Transient growth in a non-rotating shear flow*

Consider the evolution of linear shearing waves in a non-rotating, three-dimensional, incompressible shearing sheet. The wave amplitudes $(\tilde{\boldsymbol{v}}, \tilde{\psi})$ of the velocity and modified pressure perturbations satisfy the equations (see lectures)

$$\begin{aligned}
\frac{\mathrm{d}\hat{v}_x}{\mathrm{d}t} \qquad\quad &= -\mathrm{i}k_x\hat{\psi}, \\
\frac{\mathrm{d}\hat{v}_y}{\mathrm{d}t} - S\hat{v}_x &= -\mathrm{i}k_y\hat{\psi}, \\
\frac{\mathrm{d}\hat{v}_z}{\mathrm{d}t} \qquad\quad &= -\mathrm{i}k_z\hat{\psi}, \\
\mathrm{i}\boldsymbol{k}\cdot\hat{\boldsymbol{v}} &= 0,
\end{aligned}$$

where

$$\begin{aligned}
&(\tilde{\boldsymbol{v}}, \tilde{\psi}) = (\hat{\boldsymbol{v}}, \hat{\psi}) \exp\left(-\int \nu k^2 \,\mathrm{d}t\right), \\
&k^2 = k_x^2 + k_y^2 + k_z^2, \\
&\frac{\mathrm{d}k_x}{\mathrm{d}t} = Sk_y.
\end{aligned}$$

Show that, for generic non-axisymmetric disturbances ($k_y \neq 0$), the inviscid wave amplitudes satisfy

$$\hat{v}_x \propto t^{-2}, \qquad \hat{v}_y \propto \text{constant}, \qquad \hat{v}_z \propto \text{constant}$$

in the limit $t \to \infty$.

Show that generic axisymmetric disturbances ($k_y = 0$) experience algebraic growth in the absence of viscosity. When viscosity is included, show that the kinetic energy of the disturbance is proportional to

$$(1 + T^2\cos^2\theta)\,\exp\left(-\frac{2T}{\mathrm{Re}}\right),$$

where $T = St$ is the dimensionless time measured from the instant when $\hat{v}_y = 0$, θ is the angle between the wavevector and the vertical, and $\mathrm{Re} = S/\nu k^2$ is the Reynolds number of the disturbance.

Hence show that, for large Reynolds number, an energy amplification of

$$\left(\frac{\cos\theta}{\mathrm{e}}\right)^2 (\mathrm{Re})^2 + O(1)$$

can be achieved, taking a time

$$\Delta t = \frac{\mathrm{Re}}{S} + O(\mathrm{Re})^{-1}.$$

2. *Elliptical vortices*

(a) In a two-dimensional inviscid incompressible fluid, an elliptical vortex patch of semimajor axis a, semiminor axis b and uniform vorticity ξ_0 is surrounded by an irrotational flow with velocity tending to zero as $|\boldsymbol{r}| \to \infty$. Show that the velocity field induced by the vortex causes it to rotate with angular velocity

$$\frac{ab\,\xi_0}{(a+b)^2}.$$

[This result, due to Kirchhoff, can be derived in several different ways. A standard method is as follows. Introduce elliptical coordinates (μ, ν) such that

$$x = c\cosh\mu\cos\nu, \qquad y = c\sinh\mu\sin\nu, \qquad \mu > 0, \qquad 0 \leqslant \nu < 2\pi,$$

and choose $c > 0$ such that the boundary of the vortex patch at a particular instant corresponds to the curve $\mu = \mu_0 =$ constant. Verify that the coordinates are orthogonal and have scale factors $h_\mu = h_\nu = c(\sinh^2\mu + \sin^2\nu)^{1/2}$. Solve Poisson's equation $\nabla^2\chi = -\xi$ for the streamfunction by separation of variables, choosing a solution for which the velocity is finite at the singular points of the coordinate system. Then show that the normal component of the instantaneous velocity on the boundary of the vortex is the same as that of a rigid rotation with the angular velocity specified above. Note that the normal velocity component is determined by the variation of the streamfunction along the boundary.]

(b) Determine the effect of a uniform shear flow, $\boldsymbol{u} = -Sx\,\boldsymbol{e}_y$, on the shape of an elliptical material curve, such as the boundary of an elliptical vortex patch. If the ellipse has semiaxes $a(t)$ and $b(t)$, with the former being directed at an angle $\theta(t)$ (measured in a positive sense) with respect to the y axis, show that these parameters evolve in time according to the equations

$$\frac{\dot{a}}{a} = -\frac{\dot{b}}{b} = S\sin\theta\cos\theta,$$

$$\dot{\theta} = \frac{S(b^2\cos^2\theta - a^2\sin^2\theta)}{a^2 - b^2}.$$

[In this part of the question, neglect the velocity due to the vortex itself.]

3. *Integral relations for the shearing box*

A homogeneous incompressible fluid in the the shearing sheet satisfies the Navier–Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} = -\boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla}p + \nu\nabla^2\boldsymbol{u},$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0,$$

where

$$\boldsymbol{u} = -Sx\,\boldsymbol{e}_y + \boldsymbol{v}$$

is the total velocity, $\boldsymbol{\Omega} = \Omega\,\boldsymbol{e}_z$ is the angular velocity of the frame of reference and $\Phi = -\Omega S x^2$ is the effective potential (neglecting vertical gravity). The velocity perturbation $\boldsymbol{v}$ therefore satisfies

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y}\right)\boldsymbol{v} - Sv_x\,\boldsymbol{e}_y + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + 2\boldsymbol{\Omega}\times\boldsymbol{v} = -\boldsymbol{\nabla}\psi + \nu\nabla^2\boldsymbol{v},$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0.$$

where $\psi = p/\rho$.

The *shearing box* is a rectangular domain

$$0 < x < L_x, \qquad 0 < y < L_y, \qquad 0 < z < L_z,$$

on which the following boundary conditions are applied, where f stands for ψ or any component of $\boldsymbol{v}$:

$$\begin{aligned} f(0,y,z,t) &= f(L_x,(y - SL_x t) \bmod L_y, z, t),\\ f(x,0,z,t) &= f(x,L_y,z,t),\\ f(x,y,0,t) &= f(x,y,L_z,t). \end{aligned} \tag{1}$$

Interpret these boundary conditions, and show that they are compatible with solutions in the form of shearing waves in which

$$f = \mathrm{Re}\left\{\tilde{f}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\},$$

provided that the wavevector lies on the shearing lattice

$$k_x = \frac{2\pi n_x}{L_x} + Sk_y t, \qquad k_y = \frac{2\pi n_y}{L_y}, \qquad k_z = \frac{2\pi n_z}{L_z},$$

where n_x, n_y and n_z are integers.

Let $\langle\cdot\rangle$ denote a volume average over the box. Show that

$$\left\langle\frac{\partial f}{\partial x}\right\rangle = \left\langle\frac{\partial f}{\partial y}\right\rangle = \left\langle\frac{\partial f}{\partial z}\right\rangle = 0,$$

where f is any quantity satisfying the boundary conditions (1), but not necessarily a shearing wave; this result is useful for integration by parts.

Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{v}\rangle = S\langle v_x\rangle\,\boldsymbol{e}_y - 2\boldsymbol{\Omega}\times\langle\boldsymbol{v}\rangle,$$

and deduce that the mean velocity executes an epicyclic oscillation, but if initially zero will remain so.

Show further that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\langle v_x^2\rangle\right) = 2\Omega\langle v_x v_y\rangle - \nu\langle|\boldsymbol{\nabla}v_x|^2\rangle + \left\langle\psi\frac{\partial v_x}{\partial x}\right\rangle,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\langle v_y^2\rangle\right) = -(2\Omega - S)\langle v_x v_y\rangle - \nu\langle|\boldsymbol{\nabla}v_y|^2\rangle + \left\langle\psi\frac{\partial v_y}{\partial y}\right\rangle,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\langle v_z^2\rangle\right) = -\nu\langle|\boldsymbol{\nabla}v_z|^2\rangle + \left\langle\psi\frac{\partial v_z}{\partial z}\right\rangle,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\langle|\boldsymbol{v}|^2\rangle\right) = S\langle v_x v_y\rangle - \nu\langle|\boldsymbol{\nabla}\times\boldsymbol{v}|^2\rangle.$$

Deduce that, if hydrodynamic turbulence is to be maintained (without external forcing) against viscous dissipation in a Keplerian shear flow [$S = (3/2)\Omega > 0$], then $\langle v_x v_y\rangle$ must be positive (corresponding to outward transport of angular momentum) and the pressure–strain correlation $\langle\psi\,\partial v_i/\partial x_j\rangle$ must play an important role.

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