Example Sheet 3

1. *Horseshoe orbits*

In the lectures, the concept of horseshoe orbits of material in a disc containing a satellite was introduced. Consider a disc of non-interacting particles in the local approximation:

$$\ddot{x} - 2\Omega\dot{y} = 2\Omega S x - \frac{\partial\Psi}{\partial x},$$

$$\ddot{y} + 2\Omega\dot{x} = -\frac{\partial\Psi}{\partial y},$$

with $\Psi = -GM_{\rm s}\left(x^2 + y^2 + b^2\right)^{-1/2}$. Here, b is a smoothing length. An integral of motion is given by $\epsilon = (\dot{x}^2 + \dot{y}^2)/2 - \Omega S x^2 + \Psi$. For convenience, you may assume Keplerian rotation ($S = 3\Omega/2$).

(i) Show that for $y = 0$ and $b > 0$ there are three equilibrium points where accelerations and velocities are zero, and calculate their location. Show that for b large enough, only one equilibrium point survives. Sketch the streamlines in both cases. What would be an appropriate value for b in a thin disc?

(ii) By following the flow from an equilibrium point to infinity, calculate the width w of the horseshoe region.

(iii) Argue than an eccentric orbit far away from the satellite can be parametrized as $x = x_0 + e x_0 \sin\Omega t$. Show that, if a horseshoe turn is symmetric (i.e. it takes a particle from x_0 to $-x_0$ and vice versa), no eccentricity is excited.

(iv) Consider a stream of particles in the horseshoe region at distance $x = x_0 > 0$ far away from the satellite ($y > 0$). Show that a stream of particles of width δx, surface density $\Sigma(x_0)$, induces a mass flow past the satellite $\delta\dot{M} = \Sigma(x_0)v_{y0}(x_0)\delta x$. Argue that the resulting torque onto the satellite is $\delta\Gamma = \Omega x_0 \delta\dot{M}$. Show that the torque due to a stream originating at $x = -x_0$ is $\delta\Gamma = -\Omega x_0 \Sigma(-x_0) v_{y0}(x_0)\delta x$. Now take $\Sigma = \Sigma_0(1 + kx)$. Integrate over all streams making a horseshoe turn for $x_0 > 0$ to get the total torque from one side. Then show that the total torque is given by

$$\Gamma = \frac{3}{4}k\Sigma_0\Omega^2 w^4.$$

(v) Consider the y-component of the equation of motion in conservative form:

$$\frac{\partial}{\partial t}(\Sigma p_y) + \boldsymbol{\nabla}\cdot(\Sigma p_y \mathbf{u}) = -\Sigma\frac{\partial\Psi}{\partial y}.$$

Integrate over a rectangular region $R : (|x| < x_{\max}, |y| < y_{\max})$. For R large enough, argue that the right-hand side is the total torque applied by the satellite on the disc. Show that, in a steady state, the total torque on the satellite is given by

$$\Gamma = \int_{-x_{\max}}^{x_{\max}} \Sigma p_y v_y dx \Big|_{-y_{\max}}^{y_{\max}} . \tag{1}$$

Show that for initial surface density $\Sigma_i = \Sigma_0(1 + kx)$ the total torque is the same as found under (iv).

[*Hint: To arrive at (1), use the divergence theorem. Assume the only velocity perturbations are due to horseshoe turns. Note that for material that has undergone a horseshoe turn,* $\Sigma(x) = \Sigma_i(-x)$.]

2. *Magnetorotational dispersion relation*

Consider the magnetorotational dispersion relation derived in the lectures,

$$\left[(\omega + \mathrm{i}\nu k^2)(\omega + i\eta k^2) - \omega_{\mathrm{a}}^2\right]^2 - 2\Omega(2\Omega - S)(\omega + \mathrm{i}\eta k^2)^2 - 2\Omega S\omega_{\mathrm{a}}^2 = 0,$$

and assume that $\kappa^2 = 2\Omega(2\Omega - S) > 0$, $\nu > 0$ and $\eta > 0$, with $\nu \neq \eta$ in general.

(i) As the parameters are varied, instability first sets in at a bifurcation where $\mathrm{Im}(\omega)$ passes through zero. If also $\mathrm{Re}(\omega) = 0$ at this point, it is a stationary bifurcation. Otherwise it is an oscillatory bifurcation.

By considering the dispersion relation for a real value of ω, but without solving it, show that an oscillatory bifurcation cannot occur, and deduce that the instability always sets in as a monotonically growing mode.

[*Hint: Solve the imaginary part of the dispersion relation for* ω^2 *and substitute into the real part.*]

(ii) Show that the magnetorotational instability can occur even in the limit of very large viscosity, $\nu \to \infty$, provided that the magnetic diffusivity η is sufficiently small and Chandrasekhar's criterion $2\Omega S > \omega_{\mathrm{a}}^2$ is satisfied.

[*Hint: Consider the condition for marginal stability, i.e. a stationary bifurcation, in this limit, for fixed* k *and* ω_{a}.]

Show that the growth rate in this limit is approximately

$$\left[(2\Omega S)^{1/2} - |\omega_{\mathrm{a}}|\right] \frac{|\omega_{\mathrm{a}}|}{\nu k^2} - \eta k^2.$$

3. *Mechanical analogue of the MRI*

In the local approximation, the dynamics of two particles of mass m connected by a spring of spring constant $k = \beta m$ is described by the equations

$$\begin{aligned} \ddot{x}_1 - 2\Omega\dot{y}_1 - 2\Omega S x_1 &= \beta(x_2 - x_1) \\ \ddot{y}_1 + 2\Omega\dot{x}_1 &= \beta(y_2 - y_2) \\ \ddot{z}_1 + \Omega_z^2 z_1 &= \beta(z_2 - z_1), \end{aligned}$$

together with similar equations in which the suffixes 1 and 2 are interchanged.

(i) Give a physical interpretation of the equations, explaining the meaning of the symbols Ω, S and Ω_z.

(ii) Assume that the quantities β, Ω, S, $\kappa^2 = 2\Omega(2\Omega - S)$ and Ω_z^2 are all positive. Show that relative motions of the two particles in the (x, y) plane proportional to $\exp(\lambda t)$ are possible, where

$$\lambda^4 + (\kappa^2 + 4\beta)\lambda^2 + 4\beta(\beta - \Omega S) = 0.$$

(iii) Determine the range of β for which instability occurs. For fixed Ω and S, find the maximum growth rate of the instability and the value of β for which this occurs. Write down the explicit form of $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ for this optimal solution.

(iv) Discuss the relation of this problem to the MRI in astrophysical discs. In the magnetohydrodynamic case, what quantity would correspond to β in the above analysis?

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