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Mathematical Tripos Part III / Part III Astrophysics Lent Term 2014

Examples Sheet 2

1. *Effect of stellar evolution on dust orbits*
For dust (of diameter D) that acts like a black body, its radiation forces can be characterised by

$$\beta = F_{\rm rad}/F_{\rm grav} = B(L_\star/M_\star)D^{-1}. \quad (1)$$

Consider dust grains orbiting a White Dwarf (WD) for which stellar evolution models show that the mass of the star remains roughly constant while its luminosity decays so that

$$L_\star = AM_\star(t+0.1)^{-\alpha}, \quad (2)$$

where t is the WD cooling age in Myr and $\alpha > 1$. Assuming the dust orbits are roughly circular, Poynting-Robertson (P-R) drag causes their semimajor axes to decay at a rate

$$\dot{a}_{\rm pr} = -CM_\star\beta a^{-1}. \quad (3)$$

For realistic stellar and grain models the constants are $\alpha = 1.18$, $A = 5.4$, $B = 0.43$ and $C = 1250$, for D in μm, a in AU, $\dot{a}_{\rm pr}$ in AU/Myr, and $L_\star$ and $M_\star$ in Solar units.

(a) For dust grains at a semimajor axis a_0 at $t = 0$, find an expression for their semimajor axes at late times, and hence show that grains larger than $0.2ABCM_\star a_0^{-2}10^\alpha/(\alpha-1)$ never reach the star due to P-R drag, and evaluate this limit for the values of the constants given above assuming $M_\star = 0.5M_\odot$.

(b) The black body approximation given in eq. (1) breaks down for grains smaller than $\sim 0.1\mu$m, for which β remains independent of size at the value expected at $D = 0.1\mu$m. Thus show that no particles of any size ever reach the star if their initial semimajor axes are larger than $\sqrt{2ABCM_\star 10^\alpha/(\alpha-1)}$.

(c) Grains with large enough β are put on unbound orbits by radiation pressure as soon as they are created in collisions. Show that no particles are removed in this way by radiation pressure after an age of $(20AB)^{1/\alpha} - 0.1$ Myr.

(d) However, the continuous change in β does mean that radiation pressure causes the semimajor axis of the dust to decrease. Assuming that the orbit remains circular, and ignoring P-R drag for now, derive an expression for the semimajor axis decay due to radiation pressure $\dot{a}_{\rm rp}$ and hence show that the grains never reach the star by this process, but end up at late times at a semimajor axis of $a_0(1 - ABD^{-1}10^\alpha)$.

(e) Determine the ratio $\dot{a}_{\rm pr}/\dot{a}_{\rm rp}$ and thus describe in which regions of parameter space the dust semimajor axis evolution is dominated by P-R drag or by changes in radiation pressure.

2. *Vertical motion near Lagrange equilibrium points*

Linear stability analysis can be used to derive equations of motion for small perturbations from the Lagrange equilibrium points in the circular restricted three body problem. Typically it is assumed that motion is in the orbital plane of the binary (i.e., $z = 0$). This question considers the effect of a perturbation that may also be in the vertical direction.

(a) Write the equations of motion for the circular restricted three body problem for reduced masses $\mu_1 = \frac{m_1}{m_1+m_2}$ and $\mu_2 = \frac{m_2}{m_1+m_2}$ in terms of the coordinate system x, y, z that is centred on the centre of mass of the binary system and co-rotating with the mean motion n of the binary orbit so that x points to the secondary. Consider a small displacement (X, Y, Z) from an equilibrium point denoted by subscript 0, and use a Taylor expansion write the full equations for perturbed motion using the notation where e.g. $U_{xy} = \left(\frac{\partial^2 U}{\partial x \partial y}\right)_0$ (i.e., the value of the partial derivative at the equilibrium point), and

$$U = 0.5n^2(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} \tag{4}$$

is the pseudo-potential where r_1 and r_2 are the distances of the particle from the primary and secondary mass respectively.

(b) Find expressions for $\frac{\partial^2 U}{\partial x \partial z}$ and $\frac{\partial^2 U}{\partial y \partial z}$ and thus show that the evolution of the displacement in the plane is unaffected by vertical perturbations.

(c) Find U_{zz} and solve for the evolution of vertical perturbations.

(d) Show that vertical oscillations for the $L2$ point, which lies at $r_2 \approx \alpha = \left(\frac{\mu_2}{3\mu_1}\right)^{1/3}$, have a period $\frac{2\pi}{\sqrt{\mu_1(4-3\alpha)}}$.

3. *Tadpole and horseshoe orbits*
The equations of motion of a test particle in the circular restricted three body problem expressed in polar coordinates in the co-rotating frame centred on the centre of mass of the binary are

$$\ddot{r} - r\dot{\theta}^2 - 2r\dot{\theta} = \partial U/\partial r, \quad (5)$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + 2\dot{r} = r^{-1}\partial U/\partial\theta, \quad (6)$$

where r is the distance from the centre of mass, and θ is the angle with respect to the line joining the centre of mass and the secondary, the pseudo-potential $U = (1-\mu)/r_1 + \mu/r_2 + 0.5r^2$, r_1 and r_2 are distances from the primary and secondary. Units are chosen such that the primary and secondary have masses $1-\mu$ and $\mu \ll 1$, respectively, and the secondary orbits the primary at unit distance at unit angular frequency. By denoting $\Delta = r - 1 \ll 1$ as the deviation of the particle's orbit from the unit circle, it can be shown that the radial and azimuthal components of the equations of motion (to leading order, and ignoring fast epicyclic motion) are

$$3\Delta + 2\dot{\theta} \approx 0, \quad (7)$$
$$\ddot{\theta} \approx -\left(\frac{3}{2}\right)\mu\frac{\partial}{\partial\theta}\left[4\sin^2(\theta/2) + \frac{1}{\sin(\theta/2)}\right]. \quad (8)$$

(a) Determine the location of the equilibrium points of equations (7) and (8), and identify these with the associated Lagrange equilibrium points.

(b) Derive the integral relation

$$\Delta^2 + \left(\frac{4}{3}\right)\mu\left[4\sin^2(\theta/2) + \frac{1}{\sin(\theta/2)}\right] = 4\mu B, \quad (9)$$

where B is a constant of integration.

(c) Equation (9) yields the shape of tadpole or horseshoe orbits depending on the value of B. Find the value of $B = B_{L4}$ that corresponds to the L_4 (and L_5) point(s).

(d) By expanding by a small offset ϵ from the L_4 point, i.e., $\theta = \theta_{L4} + \epsilon$, derive the equation of motion for the perturbation, and show that this has a period $\frac{2\pi}{\sqrt{27\mu/4}}$.

(e) Find the value of $B = B_{L3}$ that corresponds to the maximal tadpole orbit that extends to the L_3 point, and determine how close this orbit gets to the secondary, as well as its maximum radial width.

(f) For $B > B_{L3}$, the orbit is a horseshoe that encircles L_3. Find the value of $B = B_{max}$ that corresponds to the maximal horseshoe orbit that approaches the Hill sphere of the secondary at $\theta \approx (\mu/3)^{1/3}$, and determine the maximum radial width of this orbit.

(g) Optional: Derive equations (7) and (8) by expanding the potential retaining terms of order Δ, Δ^2 and μ, and by also assuming that $d/dt \ll 1$ to filter out fast epicyclic motion.

4. *Patched conic sections / resonance overlap*

Consider close encounters between a test particle and a planet of mass $M_{\rm pl}$ that orbits a star of mass $M_\star$ on a circular orbit of semimajor axis $a_{\rm pl}$. The particle's initial orbit around the star is assumed to be co-planar with that of the planet, and circular with semimajor axis a_1, until that orbit is modified by an encounter with the planet. During the encounter, it is assumed that the gravity of the star can be ignored and the particle is considered to follow a hyperbolic trajectory about the planet. This is assumed to impart a quasi-instantaneous change to the particle's velocity which translates into a change in its orbit around the star.

(a) Assuming that $\mu^{1/3} \ll \epsilon \ll 1$, where $\mu = M_{\rm pl}/M_\star$ and $\epsilon = (a - a_{\rm pl})/a_{\rm pl}$, show that the angle of deflection θ of the particle's orbit due to the encounter in the frame rotating with the planet is given by

$$\theta \approx 8\mu\epsilon_1^{-3}, \tag{10}$$

and so determine the radial and azimuthal components of the particle's velocity in the inertial frame after the encounter.

(b) Use the Tisserand criteria, that $T_{\rm pl} = (a_{\rm pl}/a) + 2\sqrt{(a/a_{\rm pl})(1 - e^2)}$ is constant, to get an expression that relates the change in semimajor axis, characterised by $\Delta\epsilon = \epsilon_2 - \epsilon_1$, to the eccentricity of the new orbit e_2.

(c) Use the post-encounter inertial velocities of the particle to determine the eccentricity of the particle's orbit after the encounter, e_2, and thus show that the resulting change in the particle's semimajor axis is given by

$$\Delta\epsilon \approx (32/3)\mu^2\epsilon_1^{-5}. \tag{11}$$

Hint: You also need to consider the particle's post encounter true anomaly.

(d) A *conjunction* occurs when the particle and planet are at the same mean longitude, i.e., $\lambda = \lambda_{\rm pl}$. Show that subsequent conjunctions are separated in mean longitude by $(4\pi/3)\epsilon^{-1}$.

(e) Determine the effect on the longitude of the next conjunction caused by the change in orbital elements from the encounter derived in equation (11). Show that this longitude is changed by more than 2π (relative to its location had the orbital elements remained unchanged by the interaction with the planet) if the particle's initial orbit has

$$\epsilon_1 < (8\mu/3)^{2/7}, \tag{12}$$

and discuss the significance of this result.

(f) Optional: First order mean motion resonances occur where the ratio of orbital periods is $j : j + 1$. Determine the radial spacing, $\delta\epsilon_j$, between neighbouring first order mean motion resonances as a function of $j \gg 1$, and so find an expression for the critical semimajor axis for which $\Delta\epsilon = \delta\epsilon_j$.

5. *Cometary dynamics / dynamical friction*

Consider two planetary embryos on circular orbits at a distance of around $a_{\rm pl}$ from a star of mass $M_\star$. One of the embryos reaches the critical mass for rapid gas accretion, and so quickly grows to become a gas giant planet of mass $M_{\rm pl}$. The remaining embryo (of mass $M_{\rm em} \ll M_{\rm pl}$) then undergoes many scattering encounters with the newly formed planet.

(a) The embryo's scattering dynamics is similar to that of comets, and so performs a random walk in specific energy $x = 1/a$ characterised by a diffusion coefficient

$$D_x \approx \left(\frac{10}{a_{\rm pl}}\right)\left(\frac{M_{\rm pl}}{M_\star}\right). \tag{13}$$

Assuming the embryo's pericentre $q \approx a_{\rm pl}$ and its eccentricity is large (i.e., $(1-e) \ll 1$), use the Tisserand criterion to show that evolution in pericentre is much slower than evolution in the parameter x.

(b) Show that the characteristic scattering diffusion time

$$t_{\rm sc} = t_{\rm per,pl}\sqrt{\frac{a_{\rm pl}}{a}}\left(\frac{M_{\rm pl}}{M_\star}\right)^{-2}, \tag{14}$$

where $t_{\rm per,pl}$ is the planet's orbital period and a the semimajor axis of the embryo.

(c) The embryo also interacts with a coplanar belt of planetesimals that are confined to a narrow torus at orbital radii in the range $r \pm \Delta$, with a vertical height also confined within $\pm\Delta$, where r is the planetesimal's apocentric distance. The total belt mass is $M_{\rm belt}$, and individual planetesimals are much less massive than the embryo. Scattering of planetesimals by the embryo results in dynamical friction causing acceleration to the embryo of

$$\dot{\boldsymbol{v}}_{\rm em} = -4\pi G^2 M_{\rm em}\Sigma_{\rm belt}(\ln\Lambda)\boldsymbol{v}_{\rm rel}/v_{\rm rel}^3, \tag{15}$$

where $\boldsymbol{v}_{\rm rel} = \boldsymbol{v}_{\rm em} - \boldsymbol{v}_{\rm p}$ is the relative velocity vector between the embryo and planetesimals, $\Sigma_{\rm belt}$ is the mass volume density of the belt, $\Lambda = b_{\rm max}v_{\rm rel}^2/(GM_{\rm em})$, $b_{\rm max}$ is the maximum planetesimal scattering distance of relevance. Use Gauss' perturbation equations to estimate the rate of change of the embryo's orbital elements at apocentre, showing that its pericentre increases at a rate

$$\dot{q} = 4\sqrt{\frac{1-e}{1+e}}\,t_{\rm per,em}|\dot{\boldsymbol{v}}_{\rm em}|, \tag{16}$$

where $t_{\rm per,em}$ is the orbital period of the embryo, while its apocentre remains fixed.

(d) Show that the interaction lasts a fraction $(4/\pi)\sqrt{\Delta/r}$ of the embryo's orbital period.

(e) The timescale over which dynamical friction decouples the embryo from future scatterings by the planet, $t_{\rm df}$, can be estimated as that for which dynamical friction changes the embryo's pericentre by the planet's Hill radius. Assuming that $\Delta/r \approx 0.1$, and considering an embryo mass of order $1M_\oplus$, show that $t_{\rm df} < t_{\rm sc}$ as long as

$$(M_{\rm belt}/M_{\rm em}) > (M_{\rm pl}/M_\star)^{1/3}(M_{\rm em}/M_{\rm pl})^{-2}. \tag{17}$$