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Mathematical Tripos Part III / Part III Astrophysics Lent Term 2014

Examples Sheet 3

1. *Radial drift with coagulation*
After settling to the midplane of a protoplanetary disk grains then drift inward due to gas drag at a rate

$$v_{\mathrm{rd}} = -\eta v_{\mathrm{k}}[\tau_{\mathrm{s}} + 1/\tau_{\mathrm{s}}]^{-1}, \tag{1}$$

where $\eta \approx (c_{\mathrm{s}}/v_{\mathrm{k}})^2$, v_{k} is the Keplerian velocity, the sound speed $c_{\mathrm{s}} \propto \sqrt{T} \propto r^{-\alpha/2}$, the dimensionless stopping time is

$$\tau_{\mathrm{s}} = (\pi/4)\rho_{\mathrm{s}} D/\Sigma_{\mathrm{g}}, \tag{2}$$

ρ_{s} is grain density and the gas mass surface density is parametrised as $\Sigma_{\mathrm{g}} = f_{\mathrm{sc}}\Sigma_{\mathrm{g0}}(r/r_0)^{-\beta}$, where $\Sigma_{\mathrm{g0}} = 1700\mathrm{g/cm}^2$, $r_0 = 1\mathrm{AU}$ and $\beta = 1.5$.

(a) Find the size (in cm) and drift rate (in AU/yr) of the fastest moving particle as a function of r (in AU) and f_{sc} if $\rho_{\mathrm{s}} = 3\mathrm{g/cm}^3$, $\alpha = 0.5$ and $c_{\mathrm{s}} = 0.6\mathrm{km/s}$ at 1AU.

(b) Settling sets the initial size of dust that has settled to the midplane to

$$D_{\mathrm{i}} = 0.5 h f_{\mathrm{dg}}\rho_{\mathrm{gm}}/\rho_{\mathrm{s}}, \tag{3}$$

where $f_{\mathrm{dg}} = 0.01$ is the dust to gas ratio, the midplane gas volume density $\rho_{\mathrm{gm}} = \Sigma_{\mathrm{g}}/(\sqrt{2\pi}h)$ and the scale height $h = rc_{\mathrm{s}}/v_{\mathrm{k}}$. Find and comment on the particle's initial dimensionless stopping time and the initial drift rate.

(c) Show that if this particle accretes all of the mass it encounters due to its radial drift, it would grow at a rate

$$\dot{D} = 0.5 f_{\mathrm{dg}}\rho_{\mathrm{gm}}|v_{\mathrm{rd}}|/\rho_{\mathrm{s}}. \tag{4}$$

Hence show that such a model would predict that by the time particles reach the sublimation radius $r_{\mathrm{sub}} \ll r_{\mathrm{i}}$ they would have grown to a size in cm of

$$D_{\mathrm{sub}} \approx 30 f_{\mathrm{sc}} r_{\mathrm{sub}}^{-7/4}. \tag{5}$$

2. *Collisional cascade with embryo stirring and gas drag*

(a) Consider a belt of planetesimals of strength $Q_{\rm D}^{\star} = 10^5$ J kg^{-1} surrounding a star of mass $M_\star$ at a distance r in AU. Within the belt are a few embedded embryos of mass $M_{\rm em}$. If the velocity dispersion in the belt is approximately the escape speed of the embryos, and assuming all objects have a density $\rho = 3000$ kg m^{-3}, what is the minimum embryo mass to ensure that collisions between equal-sized planetesimals are catastrophic?

(b) Collisions in the belt set up a cascade with a size distribution $n(D) \propto D^{-7/2}$ between the maximum planetesimal size $D_{\rm max}$ and a minimum size $D_{\rm min} \ll D_{\rm min}$. If the fractional luminosity f of the cascade is approximately the ratio of the total cross-sectional area in the cascade to the surface area of a sphere at r, show that

$$f \propto M_{\rm tot} D_{\rm min}^{-1/2} D_{\rm max}^{-1/2},$$

where $M_{\rm tot}$ is the total mass in planetesimals. Hence, assuming that the minimum grain size is set by radiation pressure for which $\beta = 1.2 \times 10^{-3} \rho^{-1} D^{-1}$, where D is in m and ρ in kg m^{-3}, and that $D_{\rm max} = 10$ km, consider the implications of the observational constraint that $f < 10^{-3}$ at 1AU on the ability of the planetesimal belt to affect the orbits of any embedded embryos.

(c) If the scale height of the planetesimal disk h is such that h/r is approximately the ratio of the embryos' escape speed to the Keplerian velocity, the collisional lifetime of planetesimals of size D is

$$t_{\rm cc}(D) = AD^{1/2},$$

where $A \propto M_{\rm em}^{11/9} D_{\rm max}^{1/2} / M_{\rm tot}$. Assume there is also gas coincident with the planetesimal belt which causes drag that removes particles on a timescale

$$t_{\rm gas}(D) = B(\tau_s + 1/\tau_s),$$

where $\tau_s = CM_{\rm gas}^{-1} D$, B and C are constants and $M_{\rm gas}$ the gas mass. Draw a figure comparing collision lifetimes with gas drag lifetimes as a function of particle size, and note how this comparison changes with $M_{\rm gas}$ and $M_{\rm tot}$. Hence show that gas drag can become important when $M_{\rm gas} > C(2B/A)^2$, in which limit the largest particles that are affected by gas drag are those with diameters $(\frac{AM_{\rm gas}}{BC})^2$, and discuss the implications for how fractional luminosity depends on $M_{\rm gas}$.

3. *Collisional lifetime of irregular satellites*

(a) Consider a planet of mass $M_{\rm pl}$ on a circular orbit around a star of mass $M_\star$ at a distance $a_{\rm pl}$. The planet is surrounded by a swarm of irregular satellites the orbits of which are approximately circular at a distance of $\eta r_{\rm H}$, where the planet's Hill radius $r_{\rm H} = a_{\rm pl}(M_{\rm pl}/3M_\star)^{1/3}$, and their inclinations are random such that they form an isotropic distribution. Show that the average relative velocity of satellite collisions is $4/\pi$ times the Keplerian velocity at $\eta r_{\rm H}$, and so can be given by

$$\langle v_{\rm rel} \rangle = 4.6 \times 10^4 \eta^{-1/2} a_{\rm pl}^{-1/2} M_{\rm pl}^{1/3} M_\star^{1/6}, \tag{6}$$

in m s^{-1}, for masses in units of $M_\odot$ and distances in AU.

(b) Collisions set up a collisional cascade with a size distribution $n(D) \propto D^{-7/2}$ below a maximum size of $D_{\rm max}$. Assuming that satellites have a density ρ and dispersal threshold $Q_{\rm D}^\star$, and that the total satellite mass $M_{\rm sat}$ is concentrated at radii $(\eta \pm d\eta) r_{\rm H}$, work out the rate at which the largest satellites undergo catastrophic collisions $R_{\rm cc}(D_{\rm max})$.

(c) Assuming that the mass loss rate due to satellite collisions is $M_{\rm sat}(t) R_{\rm cc}(D_{\rm max})$, show that the mass of satellites remaining at late times is independent of initial mass $M_{\rm sat}(0)$ but scales as

$$M_{\rm sat}(t_{\rm late}) \propto \rho M_\star^{-13/9} M_{\rm pl}^{1/9} D_{\rm max} a_{\rm pl}^{13/3} \eta^{13/3} (d\eta/\eta) {Q_{\rm D}^\star}^{5/6} t_{\rm late}^{-1}. \tag{7}$$

(d) It is possible that extrasolar planets have irregular satellites and that the dust produced in their collisions would be detectable. What physical process would you expect to truncate the collisional cascade?

4. *Isolating terms in the disturbing function*
In a coordinate system centred on the primary star of mass $M_\star$, the perturbation potential at $\boldsymbol{r}$ (the vector offset from the origin), due to a planet of mass M_p (at $\boldsymbol{r}_p$), is

$$\phi^p(r,\theta,t) = -GM_p/|\boldsymbol{r}-\boldsymbol{r}_p| + (GM_p/|r_p|^3)\boldsymbol{r}_p \cdot \boldsymbol{r}. \tag{8}$$

This question considers perturbations in the plane of the planet's orbit.

(a) Assume the planet's orbit can be described by epicyclic motion about a guiding centre that rotates around the star at a mean frequency Ω_p, with an epicyclic frequency of κ_p for radial oscillations due to the small but non-zero eccentricity of the planet's orbit, i.e., $r_p = a(1 - e\cos\kappa_p t)$. The longitude of the planet is given by $\theta_p = \Omega_p t + 2e(\Omega_p/\kappa_p)\sin\kappa_p t$. What is the rate of precession of the planet's pericentre?

(b) The perturbing potential ϕ^p can be expanded as a Fourier series

$$\phi^p(r,\theta,t) = \sum_{l=-\infty}^{\infty}\sum_{m=0}^{\infty} \phi^p_{l,m}(r)\cos\{m\theta - [m\Omega_p + (l-m)\kappa_p]t\}. \tag{9}$$

Describe the form of the components of the perturbation potential in the frame that rotates at corresponding angular frequency $\Omega_p + (l-m)\kappa_p/m$.

(c) Assume the planet's orbit is circular. Evaluate the strength of the principal m^{th} components of the potential, $\phi^p_{m,m}(r)$, for $m=0$, $m=1$ and $m>1$, in terms of Laplace coefficients defined by

$$b_s^j(\alpha) = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos j\phi \, d\phi}{(1-2\alpha\cos\phi+\alpha^2)^s}, \tag{10}$$

using the Kronecker delta function $\delta_{m,n}$ to give an expression for $\phi^p_{m,m}(r)$ that is applicable for all m. What is the physical meaning of the additional term in the $m=1$ component?

5. *Oblate planet perturbations / ring alignment*

(a) The gravitational potential experienced by a satellite orbiting an oblate planet is given by

$$V = -\left(\frac{GM_{\rm pl}}{r}\right)\left[1 - \sum_{i=2}^{\infty} J_i \left(\frac{R_{\rm pl}}{r}\right)^i P_i(\sin\alpha)\right], \tag{11}$$

where G is the gravitational constant, $M_{\rm pl}$ and $R_{\rm pl}$ are the planet's mass and mean radius, r is distance from the planet, J_i are the dimensionless coefficients that characterise the size of the non-spherical components of the potential, $P_i(\sin\alpha)$ are Legendre polynomials of degree i, α is the latitude of the satellite above the planet's equatorial plane. Give the definition of the disturbing function, and show that by taking just the second gravitational moment J_2 this may be written as

$$\mathcal{R} = -(GM_{\rm pl}/r)J_2(R_{\rm pl}/r)^2 P_2(\sin\alpha). \tag{12}$$

(b) Using the definition $P_2(x) = 0.5(3x^2 - 1)$, write $P_2(\sin\alpha)$ in terms of the orbital elements (I, Ω, ω, f) of the satellite that are referred to the equatorial plane. Hence show that the secular part of the disturbing function (i.e., that averaged over mean longitude) can be written

$$\langle\mathcal{R}\rangle = (GM_{\rm pl}/2a)J_2(R_{\rm pl}/a)^2(1 - 1.5\sin^2 I)(1 - e^2)^{-3/2}, \tag{13}$$

where a is the semimajor axis of the orbit.

(c) Lagrange's planetary equations for the variations in the satellite's mean anomaly, longitude of ascending node and argument of pericentre are

$$\dot{M} = n - \frac{1 - e^2}{na^2 e}\frac{\partial\mathcal{R}}{\partial e} - \frac{2}{na}\frac{\partial\mathcal{R}}{\partial a}, \tag{14}$$

$$\dot{\Omega} = \frac{1}{na^2\sqrt{1 - e^2}\sin I}\frac{\partial\mathcal{R}}{\partial I}, \tag{15}$$

$$\dot{\omega} = \frac{\sqrt{1 - e^2}}{na^2 e}\frac{\partial\mathcal{R}}{\partial e} - \frac{\cos I}{na^2\sqrt{1 - e^2}\sin I}\frac{\partial\mathcal{R}}{\partial I}, \tag{16}$$

where $n = \sqrt{GM_{\rm pl}/a^3}$. Show that the planet's oblateness causes the mean motion of the satellite, n_0, to be faster than Keplerian motion by a fraction

$$(n_0 - n)/n = 1.5J_2(R_{\rm pl}/a)^2(1 - 1.5\sin^2 I)(1 - e^2)^{-1.5}. \tag{17}$$

(d) Show that to lowest order in inclination $\dot{\varpi} \approx -\dot{\Omega}$.

(e) Optional: The ϵ ring of Uranus can be modelled as two coplanar ellipses coincident with its inner and outer boundaries. The ellipses have a common focus and aligned pericentres, but different semimajor axes and eccentricities ($[a_{in}, e_{in}]$ and $[a_{out}, e_{out}]$, where $e_{out} > e_{in}$). Consider the gravitational perturbation from each of these ellipses on the other. First ascertain whether it is interactions at pericentre or apocentre that are strongest. Then use Gauss' equation for pericentre precession due to a radial acceleration, $\dot{\varpi} \propto -\bar{R}\cos f$, where f is true anomaly and $\bar{R}$ the acceleration, to consider the sign of the pericentre precession induced on each ellipse. Discuss whether the rings' self-gravity might prevent the differential precession that would otherwise be expected due to the oblateness of Uranus.