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Mathematical Tripos Part III / Part III Astrophysics Michaelmas Term 2011

Examples Sheet 3

1. *Radial drift with coagulation*

After gas drag causes small $D = 10^{-5}$ (in cm) grains to have settled to the mid-plane of a protoplanetary disk and grown to $D_{\rm i}$ in diameter, it then causes their orbits to drift inward at a rate

$$v_{\rm rd} = -\eta v_{\rm k}/(\tau_{\rm s} + 1/\tau_{\rm s}), \tag{1}$$

where $\eta \approx (c_{\rm s}/v_{\rm k})^2$ is set by how much slower the gas (at temperature T) rotates around the (solar mass) star due to its radial pressure gradient than the Keplerian velocity $v_{\rm k}$, where $c_{\rm s} \propto \sqrt{T} \propto r^{-\alpha/2}$ and the dimensionless stopping time for the Epstein drag regime is

$$\tau_{\rm s} = (\pi/4)\rho_{\rm s} D/\Sigma_{\rm g}, \tag{2}$$

where $\rho_{\rm s}$ is grain density, and the gas mass surface density is parametrised as $\Sigma_{\rm g} = \Sigma_{\rm g0}(r/r_0)^{-\beta}$.

(a) Given that a fiducial disk has $\Sigma_{\rm g0fid} = 1700{\rm g/cm}^2$, $r_0 = 1{\rm AU}$, $\beta = 1.5$, $\alpha = 0.5$, $\rho_{\rm s} = 3{\rm g/cm}^3$, with $c_{\rm s} = 0.6{\rm km/s}$ at 1AU, find the size and drift rate (in AU/yr) of the fastest moving particle at 1AU in a scaled fiducial disk for which $\Sigma_{\rm g0} = f_{\rm sc}\Sigma_{\rm g0fid}$.

(b) Show that the time it takes for particles of size $D_{\rm i}$ to migrate from $r_{\rm i}$ (in AU) to the sublimation radius $r_{\rm sub}$ (in AU) is

$$800 r_{\rm i}[\tau_{\rm si}/5 + \sqrt{r_{\rm i}/r_{\rm sub}}\tau_{\rm si}^{-1}] \tag{3}$$

years, where $\tau_{\rm si}$ is the dimensionless stopping time for particles of size $D_{\rm i}$ at $r_{\rm i}$.

(c) Settling sets the initial size of dust that has settled to the midplane to be $D_{\rm i} \approx 0.5 h f_\Sigma(\rho_{\rm gm}/\rho_{\rm s})$, where $f_\Sigma = 0.01$ is the initial gas/dust ratio, and the midplane gas volume density $\rho_{\rm gm} = \Sigma_{\rm g}/(\sqrt{2\pi}h)$ where the scale height $h = r c_{\rm s}/v_{\rm k}$. Show that the time for such particles to reach the sublimation radius is independent of $f_{\rm sc}$.

(d) Show that if this particle accretes all of the mass it encounters due to its radial drift, it would grow at a rate

$$\dot{D} = 0.5 f_\rho \rho_{\rm gm}|v_{\rm rd}|/\rho_{\rm s}, \tag{4}$$

where f_ρ is the ratio of dust/gas mass volume densities in the midplane.

(e) For the scaled fiducial disk, assuming that some settling has occurred so that $f_\rho = 0.1$, show that such a model would predict that by the time particles reach the sublimation radius they would have grown to a size in cm of

$$D_f = D_i + 300 f_{\rm sc} r_{\rm sub}^{-7/4}. \tag{5}$$

2. *Debris during terrestrial planet formation*

Following the dispersal of the protoplanetary disc $\sim$ 6Myr after a Sun-like star forms, models for terrestrial planet formation find that the region at $r = 1$AU of width $dr = 1$AU contains $N \approx 20$ planetary embryos of mass $M_{\rm pl} \approx 0.1 M_\oplus$ that continue to grow into Earth mass planets through mutual collisions in the following $\sim$ 100Myr. There is ongoing debate, however, over whether there are any planetesimals present in this region at this time, with some arguing that a mass in planetesimals comparable to that in embryos is required to damp the eccentricities of the terrestrial planets through dynamical friction.

(a) Assume $\rho = 3000{\rm kg/m}^3$ and equate the collision velocity with the escape speed of the embryos. Thus assuming a planetesimal strength $Q^\star_{\rm D} = 10^5$ J kg^{-1}, show that as long as embryos are more massive than $10^{-4} M_\oplus$, collisions between equal-sized planetesimals are expected to be catastrophic and so lead to a collisional cascade.

(b) Assuming the cascade has a size distribution $n(D) \propto D^{-7/2}$ that is truncated at $D_{\rm min}$ by radiation pressure for which $\beta = 1.2 \times 10^{-3} \rho^{-1} D^{-1}$ (where D is in m), show that the fractional luminosity of the cascade is $f \approx 12 M_{\rm tot} D_{\rm max}^{-0.5}$, where $M_{\rm tot}$ is the total mass in planetesimals (in $M_\oplus$) in this region, and $D_{\rm max} \gg D_{\rm min}$ is the maximum planetesimal size (in m). *Hint: see equation 9 of examples sheet 2.*

(c) Fractional luminosities are not observed around 10-100Myr Sun-like stars at levels higher than 10^{-3}. Derive an approximate upper limit on the total planetesimal mass around these stars at 1AU assuming a maximum size 10km, and so consider the possibility that dynamical friction damps terrestrial planet eccentricities.

(d) It is not yet clear if there is a residual gas disk present following protoplanetary disk dispersal. If so, it is possible that gas drag could truncate the lower end of the collisional cascade allowing large planetesimal masses to persist without a bright dust luminosity. To assess when and how this might occur, first estimate the collisional lifetime of planetesimals of size D in the cascade, $t_{\rm cc}(D)$. To do so, make use of the assumptions made so far in the question; you may further ignore gravitational focussing and assume the scale height of the planetesimal disk (h/r) is set by the ratio of the embryos' escape speed to the Keplerian velocity. Thus show that $t_{\rm cc}(D) \approx A D^{1/2}$, where $A \approx 0.01 D^{1/2} D_{\rm max}^{1/2} / M_{\rm tot}$ yr for $0.1 M_\oplus$ embryos.

(e) For the scaled fiducial gas disk of question 1, the lifetime of particles due to gas drag is $t_{\rm gas} = r_{\rm i}/|v_{\rm rd}| = B(\tau_s + 1/\tau_s)$, where $\tau_s = \tau_0 f_{\rm sc}^{-1} D$. Draw a figure comparing collision lifetimes with gas drag lifetimes as a function of particle size, and note how this comparison changes as f_{sc} is increased. Hence show that gas drag can become important when $f_{sc} > \tau_0 (2B/A)^2$, in which limit the largest particles that are affected by gas drag are those with diameters $(\frac{A f_{sc}}{B \tau_0})^2$, and discuss the implications for how fractional luminosity depends on f_{sc}.

3. *Collisional lifetime of irregular satellites*

Consider a planet of mass $M_{\rm pl}$ that orbits a star of mass $M_\star$ at a distance $a_{\rm pl}$ on a circular orbit. The planet is surrounded by a swarm of irregular satellites, each of density ρ, with a total mass in the swarm at time t of $M_{\rm sat}(t)$ that is concentrated in a distance range $(\eta \pm d\eta/2)r_{\rm H}$ where the planet's Hill radius $r_{\rm H} = a_{\rm pl}(M_{\rm pl}/3M_\star)^{1/3}$.

(a) Assuming that the satellite orbits are approximately circular, and that their inclinations are random such that they form an isotropic distribution, find the distribution of collision velocities at $\eta r_{\rm H}$ and hence show that the average relative velocity of satellite collisions is $4/\pi$ times the Keplerian velocity at $\eta r_{\rm H}$.

(b) Noting that the Keplerian velocity of the Earth is $v_{\rm K\oplus} = 30 \times 10^3$ m s^{-1}, show that the average relative velocity of satellite collisions from part (a) is

$$\langle v_{\rm rel} \rangle = 46 \times 10^3 \eta^{-1/2} a_{\rm pl}^{-1/2} M_{\rm pl}^{1/3} M_\star^{1/6}, \qquad (6)$$

in m s^{-1}, for masses in units of $M_\odot$ and distances in AU.

(c) Collisions set up a collisional cascade with a size distribution $n(D) \propto D^{-7/2}$. Ignoring gravitational focussing, work out the rate at which the largest satellites in the distribution of diameter $D_{\rm max}$ undergo catastrophic collisions, $R_{\rm cc}(D_{\rm max})$ (for a dispersal threshold of $Q_{\rm D}^\star$).

(d) Assuming that the mass loss rate due to satellite collisions is $M_{\rm sat}(t)R_{\rm cc}(D_{\rm max})$, show that the mass of satellites remaining at late times is independent of initial mass $M_{\rm sat}(0)$ but scales as

$$M_{\rm sat}(t_{\rm late}) \propto \rho M_\star^{-13/9} M_{\rm pl}^{1/9} D_{\rm max} a_{\rm pl}^{13/3} \eta^{13/3} (d\eta/\eta) Q_{\rm D}^{\star\,5/6} t_{\rm late}^{-1}. \qquad (7)$$

(e) It is possible that extrasolar planets have irregular satellites, and that the dust produced in their collisions would be detectable. What physical process would you expect to truncate the collisional cascade?

4. *Alternative derivation of disturbing function*

In a coordinate system centred on the primary star of mass $M_\star$, the perturbation potential at $\boldsymbol{r}$ (the vector offset from the origin), due to a planet of mass M_p (at $\boldsymbol{r}_p$), is

$$\phi^p(r,\theta,t) = -GM_p/|\boldsymbol{r}-\boldsymbol{r}_p| + (GM_p/|r_p|^3)\boldsymbol{r}_p \cdot \boldsymbol{r}. \tag{8}$$

This question considers perturbations in the plane of the planet's orbit.

(a) Assume the planet's orbit can be described by epicyclic motion about a guiding centre that rotates around the star at a mean frequency Ω_p, with an epicyclic frequency of κ_p for radial oscillations due to the small but non-zero eccentricity of the planet's orbit, i.e., $r_p = a(1 - e\cos\kappa_p t)$. The longitude of the planet is given by $\theta_p = \Omega_p t + 2e(\Omega_p/\kappa_p)\sin\kappa_p t$. What is the rate of precession of the planet's pericentre?

(b) The perturbing potential ϕ^p can be expanded as a Fourier series

$$\phi^p(r,\theta,t) = \sum_{l=-\infty}^{\infty}\sum_{m=0}^{\infty} \phi^p_{l,m}(r)\cos\{m\theta - [m\Omega_p + (l-m)\kappa_p]t\}. \tag{9}$$

Describe the form of the components of the perturbation potential in the frame that rotates at corresponding angular frequency $\Omega_p + (l-m)\kappa_p/m$.

(c) Assume the planet's orbit is circular. Evaluate the strength of the principal m^{th} components of the potential, $\phi^p_{m,m}(r)$, for $m=0$, $m=1$ and $m>1$, in terms of Laplace coefficients defined by

$$b^j_s(\alpha) = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos j\phi\, d\phi}{(1-2\alpha\cos\phi+\alpha^2)^s}, \tag{10}$$

using the Kronecker delta function $\delta_{m,n}$ to give $\phi^p_{m,m}(r)$ applicable for all m. What is the physical meaning of the additional term in the $m=1$ component?

5. Oblate planet perturbations / ring alignment

The gravitational potential experienced by a satellite orbiting an oblate planet is given by

$$V = -(GM_{\rm pl}/r)[1 - \sum_{i=2}^{\infty} J_i (R_{\rm pl}/r)^i P_i(\sin\alpha)], \tag{11}$$

where G is the gravitational constant, $M_{\rm pl}$ and $R_{\rm pl}$ are the planet's mass and mean radius, r is distance from the planet, J_i are the dimensionless coefficients that characterise the size of the non-spherical components of the potential, $P_i(\sin\alpha)$ are Legendre polynomials of degree i, α is the latitude of the satellite above the planet's equatorial plane. The satellite is treated as a test particle in this question.

(a) Give the definition of the disturbing function, and show that by taking just the second gravitational moment J_2 this may be written as

$$\mathcal{R} = -(GM_{\rm pl}/r) J_2 (R_{\rm pl}/r)^2 P_2(\sin\alpha). \tag{12}$$

(b) Using the definition $P_2(x) = 0.5(3x^2 - 1)$, write $P_2(\sin\alpha)$ in terms of the orbital elements (I, Ω, ω, f) of the satellite that are referred to the equatorial plane. Hence show that the secular part of the disturbing function (i.e., that averaged over mean longitude) can be written

$$\langle\mathcal{R}\rangle = (GM_{\rm pl}/2a) J_2 (R_{\rm pl}/a)^2 (1 - 1.5\sin^2 I)(1 - e^2)^{-3/2}, \tag{13}$$

where a is the semimajor axis of the orbit.

(c) Lagrange's planetary equations for the variations in the satellite's mean anomaly, longitude of ascending node and argument of pericentre are

$$\dot{M} = n - \frac{1-e^2}{na^2 e}\frac{\partial\mathcal{R}}{\partial e} - \frac{2}{na}\frac{\partial\mathcal{R}}{\partial a}, \tag{14}$$

$$\dot{\Omega} = \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial\mathcal{R}}{\partial I}, \tag{15}$$

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial\mathcal{R}}{\partial e} - \frac{\cos I}{na^2\sqrt{1-e^2}\sin I}\frac{\partial\mathcal{R}}{\partial I}, \tag{16}$$

where $n = \sqrt{GM_{\rm pl}/a^3}$. Show that the planet's oblateness causes the mean motion of the satellite, n_0, to be faster than Keplerian motion by a fraction

$$(n_0 - n)/n = 1.5 J_2 (R_{\rm pl}/a)^2 (1 - 1.5\sin^2 I)(1 - e^2)^{-1.5}. \tag{17}$$

(d) Show that to lowest order in inclination $\dot{\varpi} \approx -\dot{\Omega}$.

(e) The inner and outer boundaries of the ϵ ring of Uranus can be described by two coplanar ellipses that have a common focus and aligned pericentres, but different semimajor axes and eccentricities ($[a_{in}, e_{in}]$ and $[a_{out}, e_{out}]$, where $e_{out} > e_{in}$). The oblateness of Uranus should make orbits at these boundaries precess with respect to each other. Discuss whether the rings' self-gravity might prevent such differential precession, by first working out how the sign of the pericentre precession induced by self-gravity depends on $e_{\rm out} - e_{\rm in}$. *Hint: Consider the mass of the ring to be contained within two wires colocated with the boundary ellipses, and use Gauss' equation for pericentre precession due to radial and tangential accelerations of* $\bar{R}$ *and* $\bar{T}$

$$\dot{\varpi} = e^{-1}\sqrt{a(1-e^2)/\mu}[-\bar{R}\cos f + \bar{T}\sin f(2 + e\cos f)/(1 + e\cos f)]. \tag{18}$$