Superfluid Vortices: Example Sheet 1

1. Dynamics of a Bose-Einstein condensate is described by the Gross-Pitaevskii equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V_{ext}(\mathbf{x})\psi + U|\psi|^2\psi, \tag{1}$$

where $\hbar$ is the Plank's constant, m is the particle mass, $V_{\rm ext}$ is an external potential (if any), and U is the effective pair interaction. The number density is $n(\mathbf{x}) = |\psi(\mathbf{x})|^2$. The total number of particles is $N = \int |\psi(\mathbf{x})|^2\, d\mathbf{x}$.
(i) Write down the Hamiltonian functional $H[\psi, \psi^*]$ for the Gross-Pitaevskii equation and derive the equation for the ground state of the condensate.
(ii) Show that a dimensionless form of Eq.(1) without the external potential, $V_{ext}(\mathbf{x}) = 0$, can be written as

$$-2i\psi_t = \nabla^2\psi + (1 - |\psi|^2)\psi. \tag{2}$$

What are the units of length and time? What is the ground state?
(iii) Write down the equation for the solitary waves moving with velocity v in the positive z-direction in a condensate described by (2) in the frame of reference in which the solitary wave is stationary. Write down the linearised equations for the disturbances of the real and imaginary parts of ψ with respect to the ground state. Assume that the linearised equations are satisfied by sinusoidal disturbances of wavenumber k and find v as a function of k.
(iv) The wave function of a vortex ring that moves with velocity u along the straight-line vortex (with winding number $\mathcal{N} = 1$) positioned along the $z-$axis can be written as

$$\psi = [R(s) + \phi(s, z)] \exp[i\theta],$$

where (s, θ, z) are cylindrical coordinates and $R(s)$ is the amplitude of the straight-line vortex. Show that the equation on $\phi(s, z)$ in the frame of reference moving with the vortex ring can be written as

$$\begin{aligned} 2\mathrm{i}u\frac{\partial\phi}{\partial z} &= \frac{1}{s}\frac{\partial}{\partial s}\left[s\frac{\partial\phi}{\partial s}\right] + \frac{\partial^2\phi}{\partial z^2} - \frac{\phi}{s^2} \\ &+ (1 - 2R^2 - R(\phi + 2\phi^*) - |\phi|^2)\phi - R^2\phi^*. \end{aligned} \tag{3}$$

2. Show that the L^2-norm (the number of particles) $N = \int |\psi|^2\, d\mathbf{x}$, where ψ satisfies the NLS equation

$$i\psi_t + \nabla^2\psi - V(|\psi|^2)\psi = 0, \tag{4}$$

is conserved, $\frac{d}{dt}N = 0$.

[*Hint: Notice, that the Hamiltonian of the NLS equation is invariant under* $\psi \to \psi\, e^{i\gamma}$, *where* γ *ia a fixed constant.*]

3. Find the condensate density profile in the vicinity of a flat non-penetrable wall.

4. Derive the dispersion relation $\omega(k)^2$ for the one-component Bose-Einstein Condensate (Bogolyubov law for the elementary excitations).

5. The Gross-Pitaevskii equation of the wavefunction $\psi(\mathbf{x}, t)$ is given by

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{x}) + \frac{4\pi\hbar^2 a}{m}|\psi|^2\right]\psi, \tag{5}$$

(a) Show that the hydrodynamical equations for the density $\rho(\mathbf{x}, t) = |\psi(\mathbf{x}, t)|^2$ and velocity

$$\mathbf{v}(\mathbf{x}, t) = [\psi^*\nabla\psi - \nabla\psi^*\psi]/2mi\rho. \tag{6}$$

are

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\mathbf{v}\rho) = 0, \tag{7}$$

$$m\frac{\partial\mathbf{v}}{\partial t} + \nabla(\delta\mu + \frac{1}{2}m\mathbf{v}^2) = 0. \tag{8}$$

What is the expression for $\delta\mu$? What is its physical meaning?
(b) Write down the equation on the density ρ_0 of the ground state. What is the ground state density in the Thomas-Fermi regime?
(c) In the Thomas-Fermi regime derive the expression for $\delta\mu$ in terms of ρ and ρ_0.

6. Find the first three terms in the power series expansions of the straight line GP vortex of the winding number 1 given by $R(r)e^{i\theta}$, where $R^2(r) = |\psi|^2$ and r is the distance from the vortex centre measured in the healing lengths in a uniform condensate (a) for small r; (b) for large r.
(c) Specify the coefficients of a Padé approximation of the straight line GP vortex in a uniform condensate in the form

$$\rho(r) = R(r)^2 = \frac{r^2(a_1 + a_2 r^2)}{1 + b_1 r^2 + b_2 r^4}, \tag{9}$$

such that $\rho \to 1$ as $r \to \infty$.

[0]Please email corrections/comments to N.G.Berloff@damtp.cam.ac.uk