Quantum Fluids: Example Sheet 1

1. Consider an ideal gas of bosons in two dimensions, contained within a two-dimensional box of volume V_{2D}.

(a) Derive the density of states $g(E)$ for this two-dimensional system.

(b) Using this result show that the number of particles can be expressed as

$$N_{ex} = \frac{2\pi m V_{2D} k_B T}{h^2} \int_0^\infty \frac{ze^{-x}}{1 - ze^{-x}}\, dx$$

where $z = e^{\mu/k_B T}$ and $x = E/k_B T$. Solve this integral using the substitution $y = ze^{-x}$.

(c) Obtain an expression for the chemical potential μ and thereby show that Bose-Einstein condensation is possible only at $T = 0$.

2. Bose-Einstein condensates are typically confined in harmonic trapping potentials

$$V_{\text{ext}} = \frac{1}{2} m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2).$$

Using the corresponding density of states:

(a) Derive the expression for the critical number of particles.

(b) Derive the expression for the critical temperature.

(c) Determine the expression for the variation of condensate fraction N_0/N with T/T_c.

(d) In one of the first BEC experiments, a gas of 40,000 Rubidium-87 atoms (atomic mass 1.45×10^{-25} kg) underwent Bose-Einstein condensation at a temperature of 280 nK. The harmonic trap was spherically-symmetric with with $\omega_r = 1130$ Hz. Calculate the critical temperature according to the ideal Bose gas prediction. How does this compare to the result for the boxed gas (you may assume the atomic density is $2.5 \times 10^{18} m^{-3}$).

3. The compressibility β of a gas, a measure of how much it shrinks in response to a compressional force, is defined as

$$\beta = -\frac{1}{V}\frac{\partial V}{\partial P}.$$

Determine the compressibility of the ideal gas for $T < T_c$.

[*Hint: Since T_c is a function of V, you should ensure the full V-dependence is present before differentiating.*]

4. (a) Using the normalization condition, determine the dimensions of the wavefunction Ψ in S.I. units (metres, kilograms, seconds).

(b) Verify that all terms of the GPE have the same dimension.

(c) Show that $g|\Psi|^2$ has dimension of energy.

5. Consider a BEC in the Thomas-Fermi limit confined within a three-dimensional spherical harmonic trap.

(a) Normalize the wavefunction, and hence determine an expression for the Thomas-Fermi radius R_r in terms of N, a_s and ℓ_r.

(b) Determine an expression for the peak density in terms of N and R_r.

(c) Find an expression for the ratio R_r/ℓ_r, and comment on its behaviour for large N.

(d) What is the energy of the condensate?

6. Derive the expression for the variational energy of a three-dimensional trapped condensate, $E(\sigma)$. Repeat in two dimensions (for a potential $V_{\rm ext}(x,y) = m\omega_r^2(x^2+y^2)/2$) and in one dimension (for a potential $V_{\rm ext}(x) = m\omega_r^2 x^2/2$). For each case plot $E/N\hbar\omega_r$ versus the variational width σ, for some different values of the interaction parameter Na_s/ℓ_r. What effect does dimensionality have on the shape of the curves? How do this change the qualitative behaviour?

7. Consider a BEC in the non-interacting limit with wavefunction

$$\psi(x,y,z) = \sqrt{n_0}e^{-x^2/2\ell_x^2}e^{-y^2/2\ell_y^2}e^{-z^2/2\ell_z^2},$$

where n_0 is the peak density and ℓ_x, ℓ_y, ℓ_z are the harmonic oscillator lengths in three Cartesian directions. The BEC is imaged along the z-direction. Determine the form of the column-integrated density $n_{CI}(x,y)$.

[*Hint:* $\int_0^\infty \frac{1}{2}\sqrt{\pi/a}$.]

8. Consider a $1D$ uniform static condensate with $V_{\rm ext}(x) = 0$. Obtain an expression for the energy E in a length L of the condensate, in terms of n_0, g and L. Now consider the condensate to be flowing with uniform speed v_0. Show that the solution satisfies the $1D$ GPE, and confirm that the velocity field of this solution is indeed $v(x) = v_0$. What is the corresponding energy for the flowing condensate, and how does it differ from the static result? What is its momentum?

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