Quantum Fluids: Example Sheet 2

Problem 1. *Mixtures of condensates.* Consider a mixture of two different species of bosons with masses m_1 and m_2. The condensate wave functions for the two components are ψ_1 and ψ_2 and the energy functional is given by

$$E = \int \frac{\hbar^2}{2m_1}|\nabla\psi_1|^2 + V_1(\mathbf{x})|\psi_1|^2 + \frac{\hbar^2}{2m_2}|\nabla\psi_2|^2 + V_2(\mathbf{x})|\psi_2|^2 + \frac{U_{11}}{2}|\psi_1|^4 + \frac{U_{22}}{2}|\psi_2|^4 + U_{12}|\psi_1|^2|\psi_2|^2, \tag{1}$$

where U_{ij} is the effective interaction potential for an atom of species i with one of species j and V_i are the external potentials acting on species i.

(i) Write down two stationary Gross-Pitaevskii equations for the equilibrium states of species 1 and 2 assuming that the interaction conserves separately the number of atoms of the two species.

(ii) For a homogeneous gas ($V_1 = V_2 = 0$) relate the chemical potentials μ_i to densities of the ground state $n_i = |\psi_i|^2$.

(iii) For the homogeneous solution to be stable, the energy must increase for deviations of the density from uniformity. Assume that the spatial scale of the density disturbances is so large that the kinetic energy term in the energy functional plays no role. Consider the change in total energy arising from small changes δn_1 and δn_2 in the ground state densities of the two components. From this, or otherwise, deduce the sufficient conditions for stability of the system in terms of U_{11}, U_{12} and U_{22}.

(iv) Now assume that the trapping potentials are isotropic and harmonic:

$$V_i(\mathbf{x}) = \frac{1}{2}m_i\omega_i^2 r^2, \quad i = 1, 2,$$

where ω_i are oscillators frequencies. Show that the density distributions in the Thomas-Fermi approximation are

$$n_1 = \frac{\mu_1}{U_{11}}\frac{1}{1 - U_{12}^2/U_{11}U_{22}}\left[1 - \frac{U_{12}\mu_2}{U_{22}\mu_1} - \frac{r^2}{R_1^2}(1 - \frac{\lambda U_{12}}{U_{22}})\right], \tag{2}$$

$$n_2 = \frac{\mu_2}{U_{22}}\frac{1}{1 - U_{12}^2/U_{11}U_{22}}\left[1 - \frac{U_{12}\mu_1}{U_{11}\mu_2} - \frac{r^2}{R_2^2}(1 - \frac{U_{12}}{\lambda U_{11}})\right], \tag{3}$$

where R_i are the radii of the condensate clouds defined as $R_i^2 = 2\mu_i/m_i\omega_i^2$ and λ is the constant you need to identify.

Problem 2. For a dark soliton, the integrals of motion are renormalized so as to remove the contribution from the background and lead to finite values,

$$N_s = \int_{-\infty}^{+\infty} (n_0 - |\psi|^2)\, dx,$$

$$P_s = \frac{i\hbar}{2}\int_{-\infty}^{+\infty}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)\left(1 - \frac{n_0}{|\psi|^2}\right)dx,$$

$$E_s = \int_{-\infty}^{+\infty}\left(\frac{\hbar^2}{2m}\left|\frac{\partial\psi}{\partial x}\right|^2 + \frac{g}{2}(|\psi|^2 - n_0)^2\right)dx.$$

Evaluate these integrals using the dark soliton solution and leaving your answers in terms of the heling length, n_0, u, c.

Problem 3. Consider a dark soliton in a harmonically-trapped condensate. Approximating the background condensate with the Thomas-Fermi profile $n(x) = n_0(1 - x^2/R_x^2)$ (for $|x| \leq R_x$ and zero otherwise) and treating the soliton depth n_d to be constant, obtain an expression for the soliton speed as a function of its position x and depth n_d. Hence obtain an expression for the turning points of its motion.

Problem 4. Consider an object of mass M moving at velocity v_i which creates an excitation of energy E and momentum $\mathbf{p} = \hbar\mathbf{k}$. Show that Landau's critical velocity, $v_c = \min(E/p)$, is equivalent to $dE/dp = E/p$. Compare Landau's critical velocity for the ideal gas (dispersion relation $E(p) = p^2/2m$) against the weakly-interacting Bose gas. Finally show that in liquid helium II, Landau's critical velocity is $v_c = 60m/s$. (Hint: assume that near the roton minimum the dispersion relation has the approximate form $E(p) = \Delta_0 + (p - p_0)^2/(2\mu_0)$ where (at very low pressure) $\Delta_0 = 1.20 \times 10^{-22} J$ is the energy gap, $p_0 = \hbar k_0 = 2.02 \times 10^{-24}$ kg m/s is the momentum at the roton minimum, $\mu_0 = 0.161 m_4$ is the effective roton mass, and $m_4 = 6.65 \times 10^{-27}$ kg is the mass of one ^{4}He atom.

Problem 5. Use the LIA to determine the angular frequency of rotation of a Kelvin wave of wave length $\lambda = 2\pi/k$ (where k is the wavenumber) on a vortex with circulation κ. (Hint: Take the position along the vortex $\mathbf{s} = (A\cos\phi, A\sin\phi, z)$ where $\phi = kz - \omega t$ and assume $A \ll \lambda$.)