

Quantum Fluids: Example Sheet 3

(1) Virial theorem

Consider the energy functional of the density $n(\mathbf{r}) = |\psi|^2$ of the GP equation

$$E(n) = \int \left(\frac{\hbar^2}{2m} |\nabla \sqrt{n}|^2 + nV_{ext} + \frac{V_0 n^2}{2} \right) d\mathbf{r}.$$

(i) Write down the GP equation for the stationary solution ψ_0 such that $\int |\psi_0|^2 d\mathbf{r} = 1$ and integrate this equation to show that the chemical potential, μ , can be found as

$$\mu = \frac{1}{N} (E_{kin} + E_{ext} + 2E_{int})$$

with appropriately defined kinetic energy E_{kin} , trapping energy E_{ext} and internal energy E_{int} .

(ii) Consider a scaling transformation

$$\psi(x, y, z) = (1 + \nu)^{3/2} \psi_0[(1 + \nu)x, (1 + \nu)y, (1 + \nu)z]$$

in the expression of the energy functional and show that

$$2E_{kin} - 2E_{ext} + 3E_{int} = 0.$$

[Note: the energy variation has to vanish for any small variation of the function ψ around the solution ψ_0 .]

(2) A vortex in a trapped cloud. In class we considered a two-dimensional problem for the condensate in a parabolic trap in TF regime, in which we neglected the trapping force in the z -direction. Assume that in fully three-dimensional problem the semi-axis, Z , of the cloud in z - direction is much greater than the healing length, l_0 , in the centre of the cloud and the density on z -axis can be approximated by $n_0(z) = n(0)(1 - z^2/Z^2)$.

(a) Show that the energy of the vortex is approximated by by

$$\mathcal{E} = \frac{4\pi\hbar^2 n(0)}{3m} Z \ln \left[0.671 \frac{R}{l_0} \right] dz$$

where the kinetic energy change due to vertical gradients is neglected and R is the size of the cloud in xy -plane.

(b) Calculate the total angular momentum of the three-dimensional cloud of the condensate in a parabolic trap with a vortex along the z - axis, assuming that the healing length is much less than the semi-axis, Z , of the cloud in z - direction.

(3) Optical lattices.

Consider a 1D BEC with attractive interactions in an optical lattice described by

$$i\psi_t + \frac{1}{2}\psi_{xx} + \epsilon \cos(2\pi x)\psi + |\psi|^2\psi = 0. \quad (1)$$

(a) Write down the ordinary differential equation on $\xi(x)$ that describes a one-parameter family of one-soliton solutions

$$\psi(x, t) = \exp(ikt)\xi(x).$$

(b) Use the variational ansatz $\xi(x) = A(k)\text{sech}(\eta(k)x)$ and write down the equations that determine the inverse width η and the amplitude A for a given value of k .

(c) What are the asymptotic forms of η and A for very small and very large k ?

(4) Nonlocal potential.

The GP equation can be modified in the spirit of a density - functional approach to describe some aspects of superfluid helium. The idea is that the interaction energy term $\sim |\psi|^4$ in the GP energy functional is replaced by the interaction energy

$$W_c(\rho) = \frac{1}{M^2} \int \left[\frac{1}{2} \int \rho(\mathbf{x}) \tilde{V}(|\mathbf{x} - \mathbf{x}'|) \rho(\mathbf{x}') d\mathbf{x}' + \frac{W_1}{2 + \gamma} \rho^{2+\gamma} \right] d\mathbf{x},$$

where γ and W_1 are given and $\tilde{V}(|\mathbf{x} - \mathbf{x}'|)$ is the two-body interaction potential that can be taken as

$$\tilde{V}(|\mathbf{x} - \mathbf{x}'|) = \tilde{V}(r) = (\alpha + \beta A^2 r^2 + \delta A^4 r^4) \exp(-A^2 r^2),$$

where A, α, β and δ are parameters that can be chosen to give excellent agreement with the experimentally determined dispersion curve.

(a) Derive the GP equation with the nonlocal potential given above.

(b) Introduce the chemical potential μ and cast the resulting nonlocal GP equation into the dimensionless form

$$-2i \frac{\partial \Psi}{\partial t} = \nabla^2 \Psi + \Psi \left[1 - \int |\Psi(\mathbf{x}')|^2 V(|\mathbf{x} - \mathbf{x}'|) d\mathbf{x}' - \chi |\Psi|^{2(1+\gamma)} \right]. \quad (2)$$

What are the nondimensional units? Write down the expression for χ . How is V related to \tilde{V} ?

(c) What is the normalisation condition on V ?

(d*) Find the dispersion relation for (2).

(e*) Show that the equation for the amplitude of the steady straight line vortex $R(r)$ is given by

$$R''(r) + \frac{1}{r} R'(r) - \frac{R(r)}{r^2} + R(r) - \chi R(r)^{2(1+\gamma)+1}$$

$$= \frac{2\pi^{3/2}}{A} R(r) \int_0^{\infty} R(r')^2 \exp(-A^2(r^2 + r'^2)) [g_0 I_0(\sigma) - g_1 I_1(\sigma)] r' dr',$$

where I_n is the modified Bessel function of order n . What are g_0 , g_1 and σ ?