Part III
Geological Fluid Mechanics
Lent 2009
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SHEET 1

CONVECTION

The number in square brackets following the question number is a rough indication of the difficulty of the question (in arbitrary units).

1. [2] The two-dimensional 'roll' velocity field

$$u = -k^{-1}W'(z)\sin kx \quad v = 0 \quad w = W(z)\cos kx$$

is a solution, for suitable $W(z)$, of the standard, linear, Rayleigh-Bénard convention problem.

Show how to superimpose rolls at right-angles, with different amplitudes and wavelengths if necessary, to obtain motions with a rectangular and a square plan form. Show how the size of the square plan form is increased if the amplitude and wavelength of the cross rolls are identical.

Show how to superimpose rolls off-set by 120° to obtain a hexagonal plan form.

2. [2] Explain what is meant by the claim that

$$Ak^2 = -\int W\left(D^2 - k^2\right)^3 W / \int W^2$$

is a restatement, in variational form, of the eigenvalue equation

$$\left[\left(D^2 - k^2\right)^3 + Ak^2\right] W = 0.$$

Prove that the claim is correct.

Use the variational statement to obtain an approximation to the critical Rayleigh number for convection between rigid, perfectly conducting boundaries by using a simple trial function for the variational form.

3. [3] Flow is a porous medium is often described by D'Arcy's law, in which the inertia of the fluid is neglected and the viscosity term

$$\mu\nabla^2 \underset{\sim}{q}$$

is replaced by

$$-\mu k^{-1} \underset{\sim}{q}$$

in the momentum equations, where k is the permeability of the medium, assumed constant. The equations of continuity and heat conduction remain unaltered in form.

Consider a porous medium bounded by two horizontal, impermeable planes separated by a distance h which is very much less than the horizontal extent of the medium. The top plane is held at a temperature T_0 and both planes are perfectly conducting. Making the Boussinesq approximation and assuming a linear relationship between density and temperature, obtain an expression for the maximum temperature of the lower plane, T_m, for which all small disturbances decay.

The space within a closed, vertical cylinder whose cross-section is a square is filled with a porous medium. The height of the cylinder is very much larger than the horizontal dimensions. Considering the vertical walls to be insulated and making the same assumptions as before, calculate the maximum decrease in vertical temperature per unit increase in height that is stable to small disturbances. Describe the form of motion that you would expect just beyond this critical temperature gradient.

4. [4] The fluid "Hupnol", which has density ρ_0 at a temperature T_0, has the unusual property that deviations of its density about ρ_0 vary as the cube of deviations of its temperature about T_0, with smaller density corresponding to larger temperature.

A laboratory experiment consists of confining some Hupnol between two horizontal planes which apply, to a good degree of approximation, stress-free boundary conditions to the Hupnol. The planes are held at the fixed temperatures $T_0 + 1/2\Delta T$ (lower plane) and $T_0 - 1/2\Delta T$ (upper plane) with $\Delta T > 0$. The distance between the planes is very much less than their horizontal extent.

Neglecting the effects of compressibility, the variation of density on the inertia and assuming that the coefficients of viscosity, thermal conductivity, etc. are constant, derive

$$\left(\partial_t - \kappa\nabla^2\right)\left(\partial_t - \nu\nabla^2\right)\nabla^2\theta = 3kg\left(\Delta T/d\right)^3 z^2 \nabla_h^2\theta$$

$$\theta = \nabla^2\theta = \nabla^4\theta = 0 \quad \left(z = \pm\frac{1}{2}d\right)$$

as the differential equation with appropriate boundary conditions which governs the temperature perturbation, θ, of any small disturbance that might be present to disrupt the equilibrium position of no motion.

Explain the concept of "exchange of stabilities" and use the differential system you have derived above to prove that "exchange of stabilities" occurs in this example.

Obtain an approximate value of the critical Rayleigh number $A = g\Delta\rho d^3/(\kappa\nu)$ below which no disturbance will grow ($\Delta\rho$ is the positive density difference between the top and bottom layers of the Hupnol) as follows. Set $\theta = \sin\pi(z + 1/2)$ and evaluate A on the condition that the differential equation is satisfied in the space orthogonal to the solution. (This is known as the Galerkan technique.) Describe briefly, without performing any explicit calculation, how improved values of the critical Rayleigh number might be obtained.

Would you expect the exact value of the critical Rayleigh number to be greater than, equal to, or less than the critical Rayleigh number obtained if water replaced Hupnol in the experiment. Why?

Describe briefly how you think the experiment could be designed in order to yield the stress-free boundary conditions.

5. [3] A viscous fluid of thermal conductivity k and depth d has a uniform heat-source distribution of strength Q per unit volume. Show that the temperature distribution in the steady state is

$$T = \frac{1}{2}(Q/k)(d-z)z + T_0$$

the maximum value (over the depth d) of which is

$$T_m = Qd^2/(8k) + T_0$$

Considering the Rayleigh-Bénard problem between two rigid conducting plates, and assuming tht unstable and neutrally-stable disturbances are nonoscillatory, show that the relationship between the Rayleigh number

$$A = kgQd^5/8\nu k\kappa$$

and the wave number k at incipient instability is determined by the differential system

$$\begin{aligned}\left(D^2 - k^2\right)\theta &= 4(1-2z)W \\ \left(D^2 - k^2\right)W &= Ak^2\theta \\ \theta = W = DW &= 0 \quad (z = 0, 1)\end{aligned}$$

Determine an estimate for the critical value of A and k.

6. [8] Consider the double-diffusive, Rayleigh-Bénard problem, heated and salted from below, between free, perfectly-conducting boundaries. Using modified perturbation theory, or otherwise, calculate the values of γ_T and γ_S in

$$\begin{aligned}N_T &\approx 1 + \gamma_T\left(R_T - R_C\right) \qquad \left(R_T \to R_C\right) \\ N_S &\approx 1 + \gamma_S\left(R_T - R_C\right) \qquad \left(R_T \to R_C\right)\end{aligned}$$

where R_C is the exchange of stabilities critical value of the thermal Rayleigh number $\left(R_C = R_S/\tau + k^6/k^2\right)$.

What does it mean that a bifurcation is supercritical or subcritical?

Calculate the conditions under which the above bifurcation is supercritical.

(You may like to read Huppert, H.E. and Moore, D.R. (1976) "Nonlinear double-diffusive convection", *Journal of Fluid Mechanics* **78**, 821-854.)

7. [1] Is it usual in turbulent plume theory to neglect the effect of the specific mass flux $Q = \int w dA$. Does the accuracy of this approximation increase or decrease with distance from the source?

Calculate the region over which it is not strictly valid for both axisymmetric and two-dimensional line plumes.

8. [4] Prepare some lecture notes on two-dimensional turbulent line plumes. The notes might include: the definitions of specific buoyancy, momentum and mass fluxes; the governing equations and their solution in simple cases; and comparison with the results for axisymmetric plumes.

9. [2] A rectangular magma chamber of height 1 km is filled with magma whose density decreases linearly from 2700 km m^{-3} at the floor to 2660 kg m^{-3} at the roof. A tectonic movement causes relatively light magma of density 2650 kg m^{-3} to be input with negligible momentum at a rate of 100 m^3 s^{-1} into the base of the chamber from a small circular vent.

   First, <u>without</u> performing any calculations, i.e. using only your intuition, describe what you think will happen.

   Now explicitly calculate what will happen.

   A theoretician (who is relatively unknowledgeable about the dynamics of turbulent plumes) decides to model the magma in the chamber by a magma of uniform density whose total mass is identical to that of the actual magma. What results does he obtain?

   A second theoretician decides that a better model would be to consider the magma to occupy two layers of equal thickness with the total mass of each layer identical to the total mass of the actual magma confined to that thickness. What does she predict will happen to the plume?