Part III - Convection and Magnetoconvection Lent Term 2017

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Examples I

1. Write down the dimensionless equations for thermal convection in a layer with stress-free and perfectly conducting boundaries, defining the dimensionless parameters (the Prandtl and Rayleigh numbers) that appear.
Give expressions in terms of these parameters for the exponential growth rate of infinitesimal disturbances with wavenumber k. when the Prandtl number is (a) very large and (b) equal to unity. In case (a) show that when $R \gg 1$ the maximum growth rate occurs for $k^2 \approx \pi^2$ and is approximately equal to $R/4\pi^2$.

2. Convection in a porous medium. In such a medium the equation of motion takes the form

$$0 = -\nabla P - \gamma \mathbf{u} + \mathbf{F}, \quad \nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{F}$ is the body force. The boundary condition on $\mathbf{u}$ at the top and bottom of the convective layer ($z = 0, h$) is $\mathbf{u} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the unit normal.
(a) Why is there only one boundary condition rather than the two for an ordinary fluid?
(b) Assuming that the body force is the buoyancy force and that the temperature obeys the equations used in lectures, non-dimensionalise the equations and show that the stability is controlled by the single parameter $R = g\alpha\Delta T h/\kappa\gamma$ (cf the notation in lectures). Assuming that the boundaries are perfectly conducting, solve the linear stability problem and find the minimum value of R for growing solutions.
(c) Now instead consider the situation where the porosity is anisotropic, so that the ith component $-\gamma u_i$ of $-\gamma \mathbf{u}$ is replaced by

$$-\gamma^{(i)} u_i \ \ \text{(no sum)}, \quad \text{where} \ \ \gamma^{(1)} \neq \gamma^{(2)} \neq \gamma^{(3)}$$

and the $\gamma^{(i)}$ are all positive.
Now repeat the stability calculation to find the condition on R for steady solutions proportional to $e^{i\ell x+imy}$. Note that R now depends on ℓ, m separately. Find the minimum value of R under the condition $\gamma^{(1)} > \gamma^{(2)}$. [In this case it is best to write the equations in component form rather than use poloidal and toroidal parts.]

3. Consider the onset of convection with poorly conducting boundaries in a layer with stress-free velocity boundary conditions ($w = D^2w = 0$, $z = 0, 1$), and thermal boundary conditions of the form $D\theta = \pm\epsilon^4\theta$, $z = 0, 1$ where $\epsilon \ll 1$. By seeking solutions proportional to $\exp(\epsilon^4\lambda t + i\epsilon\ell x)$, $R = R_0 + \epsilon^2 R_2(\ell)$, find R_0 and an expression for λ in the leading order form

$$\lambda = -\alpha + \beta R_2 \ell^2 - \gamma\ell^4$$

where α, β, γ are positive constants to be determined. Thus find the minimum value of R for growing solutions.

4. Consider the onset of convection in a layer with perfectly conducting boundaries, and $z = 0$ a rigid boundary and $z = 1$ stress-free. Find an equation in the form of an infinite series for marginal stability problem by writing $w = \sum_1^\infty w_n \sin n\pi z$ (as in lectures, but now including both odd and even n). Hence show that the critical value of R is between that for the rigid-rigid and the free-free cases.

5. Nonlinear convection between poorly conducting boundaries. In the nonlinear domain, the equation satisfied at leading order by the vertically averaged temperature $\Theta(x, y, t)$ takes the form (after scaling)

$$\frac{\partial\Theta}{\partial t} = -\alpha\Theta - \mu\nabla^2\Theta - \nabla^4\Theta + \nabla\cdot(|\nabla\Theta|^2\nabla\Theta).$$

The equations are to be solved for bounded solutions in the (x, y) plane. Consider small amplitude solutions on a square lattice. Specifically, write $\mu = \mu_0 + \epsilon^2\mu_2$, $\partial/\partial t = \epsilon^2\partial/\partial T$, and $\Theta = A(T)e^{ikx} + B(T)e^{iky}$. By applying an appropriate solvability condition at $O(\epsilon^2)$ find the coupled evolution equations for A, B and thus determine whether Rolls or Squares are the stable pattern near onset.

6. Convection on a rectangular lattice. If we consider a lattice generated by the two vectors $(k_1, 0)$ and $(0, k_2)$, where by assumption k_1, k_2 are very similar but $k_1 \neq k_2$ then some of the symmetries of the square lattice are broken. Justify the coupled evolution equations for two modes proportional to Ae^{ik_1x} and Be^{ik_2y} in the form

$$\begin{aligned}\dot{A} &= \mu_1 A - a_1|A|^2 A - b_1|b|^2 A\\ \dot{B} &= \mu_2 B - a_2|B|^2 B - b_2|A|^2 B\end{aligned}$$

and give conditions on the coefficients that ensure the stability of the three solutions

$$\begin{aligned}\text{X-Rolls:} &\quad A \neq 0, \quad B = 0\\ \text{Y-Rolls:} &\quad A = 0, \quad B \neq 0\\ \text{Mixed Mode:} &\quad A \neq 0, \quad B \neq 0\end{aligned}$$

Now specialise to the case where $k_1 = k_c$, $k_2 = k_c + \ell$ ($\ell \ll 1$) so that $\mu_2 = \mu_1 - \gamma\ell^2$ for some positive γ. Assuming that at leading order $a_1 = a_2, b_1 = b_2$, and that $b_1 > a_1 > 0$ (ensuring the stability of Rolls on the square lattice) show that Y-Rolls are unstable if $\ell^2 > (b_1 - a_1)\mu_1/b_1\gamma$.