Part III - Convection
Lent Term 2017

Examples 3

1. *3:1 resonance.*
(a) given that the critical Rayleigh number R for stress free boundaries obeys the condition $R = (k^2 + \pi^2)^3/k^2$, where k is the horizontal wavenumber, find the value of k such that $R(k) = R(3k)$.
(b) Given that k and R are close to this double bifurcation point, suppose that the temperature is of the form $\theta = (Ae^{ikx} + Be^{3ikx}) \sin \pi z + c.c. + \ldots$ and use the idea of translational symmetry to justify the following pair of evolution equations for the evolution of A and B:

$$\dot{A} = \mu_1 A + \alpha_1 B A^{*2} - a_1|A|^2 A - b_2|B^2 A,$$
$$\dot{B} = \mu_2 B + \alpha_2 A^3 - a_2|B|^2 B - b_2|A|^2 B,$$

where all coefficients are real. Show that while α_1 and $|\alpha_2|$ can be set equal to unity by scaling, the sign of $\alpha_1\alpha_2$ cannot be changed.
(c) Now suppose that $\alpha_1 = -\alpha_2 = 1$, and that the a_i, b_i are positive. Show that solutions can exist in the form of *pure modes* ($B \neq 0$, $\dot{B} = 0$, $A = 0$), *mixed modes* of two different types ($A, B \neq 0$, $\dot{A} = \dot{B} = 0$) and *travelling waves* ($\dot{A}, \dot{B} \neq 0$, $|A|, |B| = const.$). Show that the travelling wave branch of solutions joins at each end one of the mixed mode branches.

2. In magnetoconvection convection at large values of Q and thus large horizontal wavenumber the horizontally averaged temperature can evolve on the same time scale as the convective modes. When the Boussinesq symmetry is weakly broken and the convection takes the form of hexagons, the evolution of the convection is described by the following pair of equations for the convection amplitude A (complex) and the mean temperature C(real):

$$\dot{A} = \mu A + \alpha A^{*2} - AC - b|A|^2 A,$$
$$\dot{C} = s(|A|^2 - C),$$

where α, b, s are positive real parameters and μ is real.
Show that there are solutions in the form of steady hexagons with A real, and discuss their stability. Show that when s is sufficiently small these solutions can be unstable and find the region of instability.

3. *Maxwell-Cattaneo convection.* A modification of the temperature equation to compensate for the unrealistic parabolic equation $T_t = \kappa\nabla^2 T$, which changes T at all points of space instantly, leads to the pair of equations (nondimensionalised), for deviations from a static state with uniform temperature gradient:

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = w - \nabla\cdot\mathbf{q},$$

$$C\left(\frac{\partial\mathbf{q}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{q} - \mathbf{q}\cdot\nabla\mathbf{u}\right) = -\mathbf{q} - \nabla\theta,$$

where $\mathbf{q}$ is the heat flux and C is the *Cattaneo number.* it is assumed that the velocity $\mathbf{u}$ is solenoidal and that the momentum equation is the same as for ordinary convection.
(i) Verify, by taking the divergence of the second equation above or otherwise, that there is a closed system of governing equations for $\mathbf{u}$, θ and $Q = \nabla\cdot\mathbf{q}$.
(ii) solve the linearised version of these equations assuming stress-free, perfectly conducting boundary conditions. Show that both steady and oscillatory marginal solutions may be possible, and find the smallest value of C for which oscillations can become unstable at a lower value of R than steady convection.

4. The Newell-Whitehead-Segal equation (discussed in lectures in the context of the Zigzag instability) for two dimensional modulation of rolls in the x-direction takes the form, for an amplitude equation for $A = A(X, Y, t)$

$$\frac{\partial A}{\partial t} = \mu A - |A|^2 A + \left(\frac{\partial}{\partial X} + \frac{1}{2i}\frac{\partial^2}{\partial Y^2}\right)^2 A.$$

Consider a pattern with a defect, such that as $Y \to \infty$, $A \to R_+ e^{ik_+ X}$ while as $Y \to -\infty$, $A \to R_- e^{ik_- X}$. Find the relation between k_+ and k_- that makes a steady state possible. What will happen when the relation does not hold?

5. *Convection due to surface tension variations. Or, watching paint dry.* Fluid is held between a rigid boundary at $z^* = 0$ and a free surface at $z^* = d$. The lower boundary is a perfect heat conductor held at temperature $T_R + \Delta T$, where T_R is room temperature, while at $z^* = d$ the temperature obeys Newton's law of cooling, so that

$$\frac{dT}{dz^*} + \frac{B}{d}(T - T_R) = 0$$

where B is the (positive) cooling parameter. The dynamical conditions at $z^* = d$ relate the horizontal stress to variations of surface temperature so that at $z^* = d$

$$w = 0, \quad \frac{\partial^2 w}{\partial z^{*2}} = \gamma\nabla_H^{*2} T$$

where γ is another positive parameter and ∇_H^{*2} is the horizontal laplacian. The effects of gravity are neglected throughout.
(a) Show that there is a basic state in which $\mathbf{u} = 0$ and $T = T_R + T_1(z^*)$, $T_1 = \Delta T(1 - pz^*/d)$, $p = B/(1+B)$.
(b) Seeking steady small-amplitude convective solutions, and writing $\mathbf{x}^* = d\mathbf{x}$, the vertical velocity as $\kappa w/d$ and the temperature as $T_R + T_1(z) + \Delta T\theta(x, y, z)$, show that the equations of motion and boundary conditions take the form (ignoring nonlinear interactions)

$$\nabla^4 w = 0; \quad 0 = pw + \nabla^2\theta,$$
$$w = w_z = \theta = 0, \quad z = 0; \quad w = \theta_z + B\theta = w_{zz} - G\nabla_H^2\theta = 0, \quad z = 1,$$

where $G = \Delta T\gamma d/\kappa$.
(c)* Seek solutions in which $w, \theta \propto \sin kx$, in the limit $k \to 0$, with $\tilde{G} = k^2 G$ finite, and derive a relation between $\tilde{G}$ and B for solutions to exist in this limit