Part III - Convection
Lent Term 2012

Examples 4

1. 3:1 resonance.
(a) given that the critical Rayleigh number R for stress free boundaries obeys the condition $R = (k^2 + \pi^2)^3/k^2$, where k is the horizontal wavenumber, find the value of k such that R(k)=R(3k).
(b) Given that k and R are close to this double bifurcation point, suppose that the temperature is of the form $\theta = (Ae^{ikx} + Be^{3ikx}) \sin \pi z + c.c. + \ldots$ and justify the following pair of evolution equations for the evolution of A and B:

$$\dot{A} = \mu_1 A + \alpha_1 B A^{*2} - a_1|A|^2 A - b_2|B^2 A,$$
$$\dot{B} = \mu_2 B + \alpha_2 A^3 - a_2|B|^2 B - b_2|A|^2 B,$$

where all coefficients are real. Show that while α_1 and $|\alpha_2|$ can be set equal to unity by scaling, the sign of $\alpha_1\alpha_2$ cannot be changed.
(c) Now suppose that $\alpha_1 = -\alpha_2 = 1$, and that the a_i, b_i are positive. Show that solutions can exist in the form of pure modes ($B \neq 0$, $\dot{B} = 0$, $A = 0$), mixed modes ($A, B \neq 0$, $\dot{A} = \dot{B} = 0$) and travelling waves ($\dot{A}, \dot{B} \neq 0$, $|A|, |B| = const.$), and determine the range of values of μ_2/μ_1 where each type exists. Find also the regions of stability of the pure and mixed modes.

2. In rapidly rotating convection at large horizontal wavenumber the horizontally averaged temperature can evolve on the same time scale as the convective modes. When the Boussinesq symmetry is broken and the convection takes the form of hexagons, the evolution of the convection is described by the following pair of equations for the convection amplitude A (complex) and the mean temperature C(real):

$$\dot{A} = \mu A + \alpha A^{*2} - AC - b|A|^2 A,$$
$$\dot{C} = s(|A|^2 - C),$$

where α, b, s are positive real parameters and μ is real.
Show that there are solutions in the form of steady hexagons with A real, and discuss their stability. Show that when s is sufficiently small these solutions can be unstable and find the region of instability.

3. Anisotropic convection. Consider convection in an anisotropic medium, so that the nondimensional viscous and thermal diffusion terms in the equations take the form

$$\left(\frac{\partial^2}{\partial z^2}+\nu_x\frac{\partial^2}{\partial x^2}+\nu_y\frac{\partial^2}{\partial y^2}\right)\mathbf{u}, \quad \left(\frac{\partial^2}{\partial z^2}+\kappa_x\frac{\partial^2}{\partial x^2}+\kappa_y\frac{\partial^2}{\partial y^2}\right)\theta.$$

Solve the modified stability problem to find the critical Rayleigh number for modes proportional to $e^{ikx+ily}$ (assuming stress free boundary conditions), and find the mode for which the Rayleigh number is the least as a function of $\nu_x, \nu_y, \kappa_x, \kappa_y$.

4. The Newell-Whitehead-Segal equation (discussed in lectures in the context of the Zigzag instability) for two dimensional modulation of rols in teh x-direction takes the form, for an amplitude equation for $A = A(X, Y, t)$

$$\frac{\partial A}{\partial t} = \mu A - |A|^2 A + \left(\frac{\partial}{\partial X}+\frac{1}{2i}\frac{\partial^2}{\partial Y^2}\right)^2 A.$$

Consider a pattern with a defect, such that as $Y \to \infty$, $A \to R_+ e^{ik_+X}$ while as $Y \to -\infty$, $A \to R_- e^{ik_-X}$. Find the relation between k_+ and k_- that makes a steady state possible. What will happen when the relation does not hold?

5. Convection due to surface tension variations. Or, watching paint dry. Fluid is held between a rigid boundary at $z^* = 0$ and a free surface at $z^* = d$. The lower boundary is a perfect heat conductor held at temperature $T_R + \Delta T$, where T_R is room temperature, while at $z^* = d$ the temperature obeys Newton's law of cooling, so that

$$\frac{dT}{dz^*} + \frac{B}{d}(T - T_R) = 0$$

where B is the (positive) cooling parameter. The dynamical conditions at $z^* = d$ relate the horizontal stress to variations of surface temperature so that at $z^* = d$

$$w = 0, \quad \frac{\partial^2 w}{\partial z^{*2}} = \gamma \nabla_H^{*2} T$$

where γ is another positive parameter and ∇_H^{*2} is the horizontal laplacian. The effects of gravity are neglected throughout.

(a) Show that there is a basic state in which $\mathbf{u} = 0$ and $T = T_R + T_1(z^*)$, $T_1 = \Delta T(1 - pz^*/d)$, $p = B/(1 + B)$.

(b) Seeking steady small-amplitude convective solutions, and writing $\mathbf{x}^* = d\mathbf{x}$, the vertical velocity as $\kappa w/d$ and the temperature as $T_R + T_1(z) + \Delta T\theta(x, y, z)$, show that the equations of motion and boundary conditions take the form (ignoring nonlinear interactions)

$$\nabla^4 w = 0; \quad 0 = pw + \nabla^2\theta,$$

$$w = w_z = \theta = 0, \quad z = 0; \quad w = \theta_z + B\theta = w_{zz} - G\nabla_H^2\theta = 0, \quad z = 1,$$

where $G = \Delta T \gamma d/\kappa$.

(c)* Seek solutions in which $w, \theta \propto \sin kx$, in the limit $k \to 0$, with $\tilde{G} = k^2 G$ finite, and derive a relation between $\tilde{G}$ and B for solutions to exist in this limit