Sound Generation and Propagation Dr E.J. Brambley
Mathematical Tripos Part III Lent Term 2015

Examples Sheet 2

The sign convention $\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$ *is used here, with inverse Fourier transforms having a factor of* $1/2\pi$*. Please send any comments and corrections to* `ejb48@cam.ac.uk`*.*

1. Taking a source location $\boldsymbol{y} = \boldsymbol{\eta} + \boldsymbol{X}(\boldsymbol{\eta}, \tau)$, so that $\boldsymbol{\eta}$ is a label for where the source was at time $\tau = 0$, integrate the Ffowcs-Williams–Hawkings equations to show that

$$\begin{aligned}\rho - \rho_0 = {} & \frac{\partial^2}{\partial x_i \partial x_j} \int_{F(\boldsymbol{\eta})>0} \left[\frac{T_{ij}(\boldsymbol{\eta}+\boldsymbol{X})}{4\pi c_0^2 r|1-M_r|} J_3(\boldsymbol{\eta})\right]_{\tau=\tau^*} \mathrm{d}V(\boldsymbol{\eta}) \\ & - \frac{\partial}{\partial x_i} \int_{F(\boldsymbol{\eta})=0} \left[\frac{F_i(\boldsymbol{\eta}+\boldsymbol{X})}{4\pi c_0^2 r|1-M_r|} J_2(\boldsymbol{\eta})\right]_{\tau=\tau^*} \mathrm{d}S(\boldsymbol{\eta}) \\ & + \frac{\partial}{\partial t} \int_{F(\boldsymbol{\eta})=0} \left[\frac{Q(\boldsymbol{\eta}+\boldsymbol{X})}{4\pi c_0^2 r|1-M_r|} J_2(\boldsymbol{\eta})\right]_{\tau=\tau^*} \mathrm{d}S(\boldsymbol{\eta}),\end{aligned} \qquad (*)$$

where $F(\boldsymbol{y}) = f(\boldsymbol{y}, \tau)$ at time $\tau = 0$, $r(\tau) = |\boldsymbol{x} - \boldsymbol{\eta} - \boldsymbol{X}(\boldsymbol{\eta}, \tau)|$, τ^* satisfies $\tau^* = t - r(\tau^*)/c_0$, J_3 is the Jacobian for the transformation from $\boldsymbol{y}$ to $\boldsymbol{\eta}$ space, $J_2 = J_3|\boldsymbol{\nabla_y} f|/|\boldsymbol{\nabla_\eta} F|$ is effectively the surface area Jacobian for the same transformation, and

$$M_r = \frac{x_i - \eta_i - X_i}{c_0|\boldsymbol{x} - \boldsymbol{\eta} - \boldsymbol{X}|}\frac{\partial X_i}{\partial \tau}$$

is the Mach number of the source in the observer's direction. Why does the quadrupole term in $(*)$ have an extra factor of $|1 - M_r|^{-1}$ when compared with the expression given in lectures?

2. A compact rigid sphere of radius a undergoes small oscillations in the x-direction with velocity $U(t)$ with $|U(t)| \ll c_0$. What does the compact low-Mach-number result from lectures suggest about the monopole radiation in the far field? By assuming the motion locally about the sphere to be incompressible, show that the dipole radiation in the far field is given to leading order by

$$\rho - \rho_0 \sim \frac{\rho_0 a^3}{6c_0^3 r}\ddot{U}(t - r/c_0)\cos\Theta,$$

where the observer is at a distance r from the centre of the sphere at an angle Θ to the x-axis.

3. A material surface is a theoretical surface that flows with a fluid with steady velocity $\boldsymbol{U_0}$. Show that the unit normal $\boldsymbol{n}$ to the surface evolves according to

$$\frac{D_0\boldsymbol{n}}{D_0 t} + \boldsymbol{n}\cdot\boldsymbol{\nabla}\boldsymbol{U_0} = \boldsymbol{\zeta}\wedge\boldsymbol{n} + \boldsymbol{n}\big(\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\boldsymbol{\nabla}\boldsymbol{U_0})\big),$$

where $\boldsymbol{\zeta} = \boldsymbol{\nabla}\wedge\boldsymbol{U_0}$ is the vorticity of the flow. Give a physical interpretation for each term.

4. *Rapid Distortion Theory.* A steady fluid flow $\boldsymbol{U_0}(x, y, z)$ is such that $\boldsymbol{U_0} \to U_\infty \boldsymbol{e_x}$ as $x \to -\infty$ and $\boldsymbol{e_x}\cdot\boldsymbol{U_0}$ is never zero. Define the (Y, Z)-streamline to be the streamline originating at $(x, y, z) = (-\infty, Y, Z)$, given by $\boldsymbol{x_s}(x; Y, Z) = \big(x, y_s(x; Y, Z), z_s(x; Y, Z)\big)$, and define the functions $Y_s(x, y, z)$ and $Z_s(x, y, z)$ such that the point (x, y, z) is on the (Y_s, Z_s)-streamline. Define the Lighthill drift function

$$\Delta(x, y, z) = \frac{x}{U_\infty} + \int_{-\infty}^{x} \frac{1}{\boldsymbol{e_x}\cdot\boldsymbol{U_0}\Big(\boldsymbol{x_s}\big(x'; Y_s(x, y, z), Z_s(x, y, z)\big)\Big)} - \frac{1}{U_\infty}\,\mathrm{d}x'.$$

Give a physical meaning for $\Delta(\boldsymbol{x_2}) - \Delta(\boldsymbol{x_1})$ when $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ lie on the same streamline.

Now define $\boldsymbol{X}(x, y, z) = (U_\infty\Delta, Y_s, Z_s)$. Deduce that

$$\text{(i)}\quad \frac{\partial\boldsymbol{X}}{\partial t} = 0 \qquad \text{(ii)}\quad \frac{D_0\boldsymbol{X}}{D_0 t} = U_\infty\boldsymbol{e_x}. \qquad \text{(iii)}\quad \boldsymbol{X} \to (x, y, z) \text{ as } x \to -\infty.$$

Hence, show that the most general solution to $\frac{D_0 s'}{D_0 t} = 0$ is $s' = S(\boldsymbol{X} - U_\infty t\boldsymbol{e_x})$, where S is an arbitrary function of its arguments.

Show that

$$v_i = \boldsymbol{A}(\boldsymbol{X} - U_\infty t\boldsymbol{e_x}) \cdot \frac{\partial \boldsymbol{X}}{\partial x_i}, \qquad \Rightarrow \qquad \frac{D_0 \boldsymbol{v}}{D_0 t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{U_0} = \boldsymbol{\zeta} \wedge \boldsymbol{v},$$

where $\boldsymbol{\zeta} = \boldsymbol{\nabla} \wedge \boldsymbol{U_0}$ is the mean flow vorticity, $\boldsymbol{v} = (v_1, v_2, v_3)$, and $\boldsymbol{A}$ is an arbitrary function of its arguments. What is $\boldsymbol{v}$ as $x \to -\infty$? Deduce that, for a potential flow, the vortical disturbance is given by

$$u_i^{(R)} = \left[\boldsymbol{u'_\infty}(\boldsymbol{X} - U_\infty t\boldsymbol{e_x}) - \boldsymbol{e_x} \frac{U_\infty s'_\infty(\boldsymbol{X} - U_\infty t\boldsymbol{e_x})}{2c_p}\right] \frac{\partial \boldsymbol{X}}{\partial x_i}$$

where $\boldsymbol{u'_\infty}(\boldsymbol{x} - U_\infty t\boldsymbol{e_x})$ and $s'_\infty(\boldsymbol{x} - U_\infty t\boldsymbol{e_x})$ are the velocity and entropy perturbations far upstream.

[This describes how vortical turbulence is distorted by a nonuniform potential mean flow $\boldsymbol{U_0}$, provided this distortion is rapid on the time-scale of viscous and thermal diffusion. This is important for calculating the sound generated by upstream turbulence encountering flow around an object (such as a propeller), as the turbulence that hits the object will have been distorted relative to the upstream turbulence.]

5. A loudspeaker undergoes small amplitude oscillations about $y = 0$ with velocity

$$V(x) = \begin{cases} \frac{1}{a}\big(1 - |x|/a\big)\exp\{\mathrm{i}\omega t\} & |x| < a \\ 0 & |x| > a \end{cases}$$

for a given frequency ω. The loudspeaker is impermeable, and generates sound in an otherwise stationary inviscid perfect gas occupying the region $y > 0$. Find the directivity of the sound when (i) $ak_0 \ll 1$ and (ii) $ak_0 \gg 1$.

6. Given $F(k)$ analytic and nonzero in a strip $-\sigma^+ < \mathrm{Im}(k) < \sigma^-$, consider a multiplicative factorization $F(k) = F^+(k)F^-(k)$ with $F^+(k)$ analytic and nonzero for $\mathrm{Im}(k) > -\sigma^+$ and $F^-(k)$ analytic and nonzero for $\mathrm{Im}(k) < \sigma^-$. Show that there is a unique factorization (up to multiplication by a constant) such that $F^\pm$ have at most algebraic growth or decay as $|k| \to \infty$ in their respective regions, meaning $|k|^{-n} < |F^\pm(k)| < |k|^n$ for all $|k| > M$ for some M and n).

7. (**a**) Give an additive decomposition of both $\cos(z)$ and $\sin(z)/z$.

   (**b**) Give a multiplicative decomposition of $k^4 - k_0^4$ with $-\pi/2 < \arg(k_0) < 0$.

   (**c**) Use the integral formula to find an additive factorization of $F(k) = \big[(k + k_0 \cos\theta_0)\sqrt{k - k_0}\big]^{-1}$, where $\mathrm{Im}(k_0) < 0$ and the branch cut is chosen such that $\sqrt{k - k_0}$ is analytic in the upper half plane. *Hint: Close the contours in the upper half plane for both F^+ and F^-.*