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Examples Sheet 3

The sign convention $\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$ *is used here, with inverse Fourier transforms having a factor of* $1/2\pi$*. Please send any comments and corrections to* `ejb48@cam.ac.uk`*.*

1. (**a**) Write down the exact solution (in integral form) for the 2D scattering of a plane wave incident at an angle θ_0 to a rigid semi-infinite plate, observed at a distance r and angle θ from the end of the plate. Hence show that the scattered field can be expressed in the far field to leading order as

$$\phi \sim Ae^{-\mathrm{i}k_0 r}\int_{-\infty}^{\infty}\frac{e^{-u^2}\,\mathrm{d}u}{u - e^{3\pi\mathrm{i}/4}\beta} + 2\pi\mathrm{i}H(-\beta)\frac{A\gamma^-(k_0\cos\theta)}{\gamma^-(-k_0\cos\theta_0)}\exp\big\{\mathrm{i}k_0 r(\cos\theta\cos\theta_0 - |\sin\theta\sin\theta_0|)\big\},$$

where

$$A = \frac{\mathrm{sgn}(\sin\theta)k_0\sin\theta_0}{2\pi\gamma^+(-k_0\cos\theta_0)\gamma^-(k_0\cos\theta)}, \qquad \text{and} \qquad \beta = \frac{\cos\theta + \cos\theta_0}{|\sin\theta|}\sqrt{\tfrac{1}{2}rk_0}.$$

(**b**) Show that

$$I(\lambda) = \int_{-\infty}^{\infty}\frac{e^{-u^2\lambda}}{u - e^{3\pi\mathrm{i}/4}\beta}\,\mathrm{d}u \qquad \Rightarrow \qquad \frac{\mathrm{d}I}{\mathrm{d}\lambda} - \mathrm{i}\beta^2 I = -e^{3\pi\mathrm{i}/4}\beta\sqrt{\frac{\pi}{\lambda}},$$

and hence that

$$I(1) = \mathrm{i}\pi\mathrm{sgn}(\beta)e^{\mathrm{i}\beta^2}\mathrm{erfc}\big(e^{\mathrm{i}\pi/4}|\beta|\big), \qquad \text{where} \quad \mathrm{erfc}(z) = 1 - \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}\,\mathrm{d}t.$$

(**c**) Now let $r \to \infty$ and $\cos\theta \to -\cos\theta_0$ such that β remains fixed. In this limit, show that

$$\mathrm{i}k_0 r(\cos\theta\cos\theta_0 - |\sin\theta\sin\theta_0|) = -\mathrm{i}k_0 r + \mathrm{i}\beta^2 + O\big(r^{-1/2}\big),$$

and hence that

$$\phi \sim \mathrm{i}\pi A e^{-\mathrm{i}k_0 r + \mathrm{i}\beta^2}\mathrm{erfc}\big(e^{\mathrm{i}\pi/4}\beta\big).$$

Interpret this as showing that the geometric optic pole “turns on” smoothly over an angle of order $(rk_0)^{-1/2}$; this solution is the Fresnel field mentioned in lectures.

2. Consider a thin plate at $y = 0, x < 0$ undergoing small amplitude oscillations with normal velocity $V_0\exp\{\mathrm{i}\omega t - \mathrm{i}qx\}$. Find an expression for the generated fluid perturbation and calculate the sound radiated to the far field. *Hint: take* $\mathrm{Im}(\omega) < 0$ *and* $\mathrm{Im}(q) > 0$*, apply the Wiener–Hopf technique, then let the imaginary parts tend to zero.*

3. A loudspeaker causes a velocity in the x-direction of $U(t)$ at $x = 0$, with $\varepsilon = \max U/c_0 \ll 1$. By considering the $O(\varepsilon)$ and $O(\varepsilon^2)$ terms in the asymptotic expansion of the x-velocity $u(x,t)$, show that

$$u(x,t) = U(\tau) + x\left[\frac{\gamma+1}{2c_0^2}U(\tau)U'(\tau) + \frac{\frac{4}{3}\mu + \mu_B + \frac{\gamma-1}{c_p}\kappa}{2\rho_0 c_0^3}U''(\tau)\right] + O\big(U^3, \mu U^2, \kappa U^2\big),$$

where $\tau = t - x/c_0$. Hence show that this asymptotic expansion breaks down when x is of the order of λ/ε, where λ is a typical wavelength.

4. Solve the inviscid Burgers' equation with initial condition $f(0,\theta) = (1 - |\theta|)$ for $|\theta| < 1$ and $f(0,\theta) = 0$ otherwise. At what value of Z does a shock form, and what is the solution for larger Z than this?

5. A travelling wave solution to the viscous Burgers' equation is a solution of the form $f(Z,\theta) = q(\theta + VZ)$ with V constant. If $f(0,\theta) \to q_-$ as $\theta \to -\infty$ and $f(0,\theta) \to q_+$ as $\theta \to +\infty$, find all possible travelling wave solutions. What happens as the dissipation $\alpha \to 0$.

6. Let $f(0,\theta) = A\delta(\theta)$. By using the Cole–Hopf transform of the viscous Burgers' equation, find $f(Z,\theta)$, and hence show that, for $2\alpha/A \gg 1$,

$$f(Z,\theta) \sim \frac{A}{\sqrt{4\pi\alpha Z}}e^{-\theta^2/4\alpha Z}.$$

What is the corresponding result for $2\alpha/A \ll 1$?

7. Explain the relation between the inviscid Burgers' weak shock theory equal area rule and the viscous Burgers' steepest descent equal area rule,

$$\frac{\mathrm{d}}{\mathrm{d}Z}\int_{\theta_1}^{\theta_2} f(Z,\theta)\,\mathrm{d}\theta = \left[\frac{1}{2}f(Z,\theta)^2\right]_{\theta_1}^{\theta_2} \qquad \text{and} \qquad \frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2} f(0,\phi)\,\mathrm{d}\phi = \frac{1}{2}\Big(f(0,\phi_2)+f(0,\phi_1)\Big).$$

8. A rectangular waveguide in the x-direction has width W and height H. The two side walls and the bottom wall are hard and do not move, while the top boundary is "soft", meaning it cannot support a pressure different from ambient, and hence $\tilde{p} = 0$ there. There is a subsonic uniform mean flow of velocity U_0 along the waveguide.

(**a**) Solve for the wave modes in the waveguide, and hence show that waves can propagate along the waveguide if and only if

$$\omega > \frac{\pi}{2H}\sqrt{c_0^2 - U_0^2}.$$

(**b**) Now consider a duct with width $W = 2H/\sqrt{5}$ and with no mean flow ($U_0 = 0$). At $x = 0$ a plate blocking the whole waveguide cross-section and hinged on one side of the waveguide flaps with small amplitude and frequency $\omega = \sqrt{3}\pi c_0/W$. Sufficiently far from the plate that evanescent waves can be neglected, find the radiated pressure. Roughly how far away from the plate is sufficiently far?

9. Consider a uniform flow of velocity U_0 along a cylindrical duct of radius a with a locally-reacting boundary of impedance $Z(\omega)$. By considering $r = a(1-\varepsilon y)$ and suitably scaling ω, k and m, find the leading-order dispersion relation for "surface modes" localized close to the boundary at high frequencies. For a given frequency ω, how many of these modes are there?