

In these examples, ‘BN’ refers to the background lecture notes for the course.

1. Rossby modes in an idealized ‘ocean basin’.

- (i) Show that the boundary-value problem for the normal-mode solutions $e^{-i\omega t} f(x, y)$ of $\nabla^2 \psi_t + \beta \psi_x = 0$, with $\psi = 0$ on the boundary $\partial\mathcal{R}$ of an arbitrary region \mathcal{R} in the x, y plane, is solved by

$$f = e^{-i\beta x/2\omega} g(x, y), \quad \omega = \pm\beta/2\lambda,$$

where $g(x, y)$ is an eigenfunction and λ an eigenvalue of the membrane problem $\nabla^2 g + \lambda^2 g = 0$ with $g = 0$ on $\partial\mathcal{R}$. [Note that, in contrast with classical small-vibration problems, $f(x, y)$ is necessarily complex-valued, and that the Rossby modes involve westward-moving nodal lines, as well as the (stationary) nodes, if any, of $g(x, y)$.]

- (ii) Solve the reflection problem for simple plane Rossby waves of the form $\psi = \sin ly e^{ikx - i\omega t}$ incident on a straight ‘north–south’ boundary $x = 0$ [see BN p. 130].
 (iii) Solve the basin problem in the case where \mathcal{R} is a rectangular region $0 \leq x \leq a$, $0 \leq y \leq b$, and show that the solution can be regarded as a case of (ii).

2. [From the 1997–8 examination] Starting from the full shallow-water equations, for uniform undisturbed depth h_{00} say, derive the quasi-geostrophic theory for a shallow-water system with constant Coriolis parameter f . Explain carefully the scaling assumptions and scaling arguments used. *Briefly* comment on the occurrence of only one time derivative $\partial/\partial t$ in the quasi-geostrophic theory, as contrasted with three in the full shallow-water equations, referring to what types of motion are described by the latter and not the former.

Uniform flow $\mathbf{u} = (U, 0)$ in a quasi-geostrophic shallow-water system with uniform potential vorticity f/h_{00} encounters topography that generates a steady flow pattern with disturbance streamfunction $\psi'(x, y)$, representing the topographically-induced departure from uniform flow. Show that a disturbance streamfunction $\psi' \propto \exp\{-(x^2 + y^2)/a^2\}$, where a is a constant length scale, provides a solution to this problem in the case where the topography $b(x, y)$ has the circularly symmetric shape

$$z = b(x, y) = \epsilon \left\{ \frac{4(a^2 - x^2 - y^2)}{a^2} + \frac{a^2}{L_D^2} \right\} \exp \left\{ \frac{-(x^2 + y^2)}{a^2} \right\},$$

where ϵ is a sufficiently small constant and L_D the appropriate Rossby length. Discuss (a) how small ϵ must be in order for the solution to represent a self-consistent use of quasi-geostrophic theory, and (b) how small ϵ must be if the flow over the topography is to have no closed streamlines.

Discuss briefly the extent to which the appearance of closed streamlines in the solution might, or might not, affect the validity of the solution as a model of physical reality.

3. The equatorial waveguide. Find the equatorially trapped solutions to the shallow-water equations in the ‘equatorial β -plane’ model, i.e. flat-earth theory but with the Coriolis parameter $f = \beta y$ ($\beta = \text{constant}$; $y = 0$ is the equator). Linearize the equations about a state of rest relative to the earth with constant layer depth h_0 , and look for waveguide modes in which the disturbance velocity,

pressure and buoyancy fields take the form $\text{func}(y) \exp(ikx - i\omega t)$, with different functions of y for the different fields. You may find it convenient to begin by making the equations dimensionless with respect to the length scale $L = (c/\beta)^{\frac{1}{2}}$ and the corresponding time scale L/c , where c is the gravity-wave speed. Show first that there is an eastward-travelling equatorially trapped mode with northward velocity component $v' = 0$ everywhere and with u' and ζ proportional to $\exp(-\frac{1}{2}y^2)$, in dimensionless variables, and with $\omega = ck$.

[This is the equatorial Kelvin wave; note that its propagation is nondispersive. You might like to ponder why this is so. Note that the x -momentum and mass-conservation equations form a closed sub-system of equations not involving f .]

Show further that modes with $v \neq 0$ have $v = V(y) \exp(ikx - i\omega t)$ where the real-valued function $V(y)$ satisfies the harmonic-oscillator case of Schrödinger's equation, namely

$$\frac{d^2V}{dy^2} + (\omega^2 - k^2 - \frac{k}{\omega} - y^2)V = 0$$

in dimensionless variables. Deduce that all such equatorially trapped modes have the structure $V \propto H_n(y)e^{-\frac{1}{2}y^2}$ and satisfy the dispersion relation

$$\omega^2 - k^2 - \frac{k}{\omega} = 2n + 1 \quad (n = 0, 1, 2, 3, \dots),$$

where the Hermite polynomials $H_0 = 1$, $H_1 = 2y$, $H_2 = 4y^2 - 2$, $H_3 = 8y^3 - 12y$ etc. You may use the fact that the Hermite polynomials are defined such that $H_n'' - 2yH_n' + 2nH_n = 0$.

Show that the dispersion relation factorizes when $n = 0$, and that only the root for which

$$\omega - k - \frac{1}{\omega} = 0$$

corresponds to an equatorially trapped solution. [This is known as the Rossby-gravity wave.]

[This theory can also be applied (and more strongly justified) in the case of a continuously stratified, quasi-hydrostatic fluid with buoyancy frequency N , by separation of variables. The same horizontal structure is found; the only change is that c is replaced by the internal gravity horizontal phase speed, e.g. N/m in the Boussinesq, quasi-hydrostatic, constant- N case with vertical structure e^{imz} . Note that, like simple internal gravity waves, *all* such waves have vertical phase velocities ω/m that are downward whenever the group velocity is upward, and vice versa; this follows simply from the nondimensionalization. These waves, especially the Kelvin wave, make a significant contribution to the quasi-biennial oscillation (QBO) in the real atmosphere.]

4. Consider the two-dimensional vortex-dynamical system

$$\left. \begin{aligned} Dq/Dt &= 0, \\ \psi &= \nabla^{-2}(q - q_0) = \nabla^{-2}(\Delta q), \text{ say,} \end{aligned} \right\} \quad (1)$$

with $u = -\psi_y$, $v = \psi_x$, $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y$, representing homogeneous (unstratified) fluid flow in an unbounded domain, the xy -plane. State the superposition principle for the inversion operator ∇^{-2} , by which one can deduce the velocity field due to a complicated pattern Δq of vorticity anomalies from a knowledge of the velocity fields due to simpler Δq patterns.

Find the velocity fields that correspond, via inversion, to the vorticity anomalies

$$\Delta q = \begin{cases} C, & r^2 = x^2 + y^2 < a^2 \\ 0, & r^2 > a^2 \end{cases} \quad (2)$$

and

$$\Delta q = \begin{cases} C, & |y| < b \\ 0, & |y| > b, \end{cases} \quad (3)$$

where C is a positive constant, and explain why both cases represent steady solutions of (1) in the case $q_0 = \text{constant}$. (You may remove any ambiguities in the inversions by making distant values of $(u^2 + v^2)$ minimal on average, i.e. zero as $r \rightarrow \infty$ for (2), and equal as $y \rightarrow \pm\infty$ for (3).)

Now assume that small disturbances to the steady flow corresponding to (3) grow to finite amplitude in the manner illustrated by the numerical simulation on BN p. 97, the early stages being dominated by an undular disturbance whose x -wavenumber k has the value corresponding to the maximum growth rate predicted by small-amplitude instability theory. This value is $k = 0.797/2b$ and you may take it as given. Assume further that the whole process leads to a final state that can be idealized, ignoring the small-scale detail, as a row of discrete circular vortices containing all the fluid with nonzero Δq . Thus the final state is assumed to have a vorticity distribution given by the periodic extension of (2) in the x direction, with spatial period $2\pi/k$ and a suitably chosen value of a . Show by equating areas that this value is given by

$$0.797a^2 = 8b^2 \quad (\text{so that } 2a = 6.3b < 2\pi/k = 15.8b).$$

Write down an expression for the associated velocity field.

5. In the last problem, explain how the superposition principle implies the existence of two different methods of calculating the *mean* velocity field $\bar{u}(y)$, under the foregoing assumptions, where ‘mean’ signifies an average with respect to x at fixed y .
- (a) Use whichever you think is the easiest of the two methods to deduce that $\bar{u} = \text{constant}$ for $|y| > a$, that \bar{u} is continuous across $|y| = a$, and that

$$\bar{u}(y) = -2b\pi^{-1}C \left\{ a^{-2}y(a^2 - y^2)^{\frac{1}{2}} + \arcsin(y/a) \right\} \quad \text{for } |y| < a.$$

Show (either by direct calculation or by reasoning from the superposition principle) that the derivative $d\bar{u}/dy$ is also continuous across $|y| = a$. Sketch the function $\bar{u}(y)$.

- (b) Deduce from your sketch that the kinetic energy of the final mean flow $\bar{u}(y)$, in the chosen frame of reference, must have been reduced in comparison with the initial kinetic energy.
- (c) Consider the x -averaged fields in a problem that is similar except for having a constant background vorticity gradient β initially. Assume that a breaking Rossby wave perfectly mixes the vorticity in a zone $|y| < b$, so that the x -averaged vorticity gradient is still β for $|y| > b$ but is zero for $|y| < b$. Invert the x -averaged vorticity change $\Delta\bar{q}(y)$ to obtain the the corresponding x -averaged velocity change $\Delta\bar{u}(y)$. Sketch the shapes of the profiles of $\Delta\bar{q}(y)$ and $\Delta\bar{u}(y)$. Show that the total (y -integrated) momentum change is proportional to

$$\int_{-\infty}^{\infty} \Delta\bar{u}(y) dy = \int_{-\infty}^{\infty} y \Delta\bar{q}(y) dy = -\frac{2}{3} \beta b^3.$$

[*The minus sign signals a chirality or handedness (one-way-ness) in the transport of momentum and angular momentum by Rossby waves. This is fundamental to the ratchet-like way

in which breaking Rossby waves drive the stratospheric mean circulation by “gyroscopic pumping”. In a longitudinally-averaged description, the wave-breaking systematically pushes air westward, and Coriolis forces systematically turn it poleward. (This results in a systematic mechanical pumping action. It is why man-made chlorofluorocarbons, for instance, are pulled up into the tropical stratosphere, then photolyzed by solar ultraviolet, and then pushed poleward and back downward — a process with a timescale of several years. For further discussion see www.atm.damtp.cam.ac.uk/people/mem/papers/ECMWF/ecmwf05.html.)*

6. [This is a shallow-water counterpart of question 5(c), from the 2006 examination.] Write down the shallow-water momentum and mass-conservation equations for a layer of homogeneous fluid of depth $h(x, y, t)$ in a frame of reference rotating with constant angular velocity $(0, 0, \frac{1}{2}f)$. Allow for a sloping bottom boundary $z = b(x, y)$, taking care to distinguish between the layer depth h and the free-surface elevation $\zeta(x, y, t)$.

Derive the equation for the vertical component of absolute vorticity, $q_a(x, y, t)$. Deduce Rossby’s exact potential-vorticity conservation theorem, explaining why it is h and not ζ that enters into the expression for the potential vorticity. Why is there no term in $w\partial/\partial z$?

Assume now that $q \ll f$ where q is the relative vorticity, $q = q_a - f$, and that $\zeta \ll h_{00}$ and $b \ll h_{00}$ where $h = h_{00} + \zeta - b$ with h_{00} constant. Derive an approximate expression for the potential vorticity, correct to the first order of small quantities, in which the leading term is f/h_{00} .

For small Rossby numbers show that the horizontal velocity field can be represented by a streamfunction $\psi(x, y, t)$, to be specified. Show that the approximate potential vorticity just derived is then proportional to

$$f + \frac{f}{h_{00}}b + \nabla_H^2\psi - \frac{\psi}{L_D^2} \quad (*)$$

where L_D is a constant to be specified. Specify also the meaning of the symbol ∇_H^2 .

Consider a thought-experiment in which the initial state is one of relative rest, with $\zeta = 0$ everywhere, and with a lower boundary specified by

$$b = b(y) = \begin{cases} \epsilon a, & y > a \\ \epsilon y, & |y| < a \\ -\epsilon a, & y < -a \end{cases}$$

where a and ϵ are positive constants with $a\epsilon/h_{00} \ll 1$. Assume that a breaking Rossby wave perfectly mixes the potential vorticity in the zone $|y| < a$, so that the x -averaged potential vorticity is unaffected outside $|y| < a$ but now has the constant value f/h_{00} within $|y| < a$. Sketch the change in (*) as a function of y . Deduce that within $|y| < a$ the resulting velocity profile $u(y)$ is given by

$$u = \frac{\epsilon f L_D^2}{h_{00}} \left\{ -1 + \left(1 + \frac{a}{L_D} \right) \exp\left(\frac{-a}{L_D}\right) \cosh\left(\frac{y}{L_D}\right) \right\}.$$

In the case $L_D \gg a$, show further that

$$u = \frac{\epsilon f L_D^2}{h_{00}} \left\{ \frac{y^2 - a^2}{2L_D^2} + O\left(\frac{a}{L_D}\right)^3 \right\} \quad (|y| < a).$$

In the case $L_D \ll a$, sketch the profiles of $u(y)$ and $\zeta(y)$, for $|y| < 2a$.

Please email any corrections/comments to mem@damtp.cam.ac.uk