Nonlinear Continuum Mechanics
Problem Sheet 2

1. Repeat the reasoning given for elementary thermodynamics, employing the Lagrangian framework: integrate over the undeformed configuration, employing nominal stress $\mathbf{P}$ and an analogous nominal heat flux vector $\mathbf{q}^0$, defined so that the heat flux across a surface element d$\mathbf{S}$ is equal to $\mathrm{d}\mathbf{S}^0 \cdot \mathbf{q}^0$. In particular, obtain the energy balance equation

$$\rho_0 \dot{u} = \rho_0 r + P_{Ii}\dot{F}_{iI} - \mathrm{Div}\mathbf{q}^0$$

and also the form

$$\rho_0 \theta \dot{\eta} = f_r^0 \dot{\xi}_r + \rho_0 r - \mathrm{Div}\mathbf{q}^0,$$

where $f_r^0 = -\rho_0 \partial u/\partial \xi_r$.

2. Develop explicitly the equations of thermoelasticity in Lagrangian coordinates, assuming the free energy function ψ given in the quadratic approximation

$$\rho_0 \psi(\mathbf{F}, \theta) = \tfrac{1}{2} E_{IJ} C_{IJKL} E_{KL} - \beta_{IJ} E_{IJ}(\theta - \theta_0) - \tfrac{1}{2} C_v (\theta - \theta_0)^2.$$

[Here, E_{IJ} could be any strain measure, but take it in particular to be the Green measure $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$.]
In the case of small deformations, so that $\mathbf{X} \to \mathbf{x} = \mathbf{X} + \mathbf{u}$, with $\|\partial \mathbf{u}/\partial \mathbf{X}\| \ll 1$, the distinction between Lagrangian and Eulerian coordinates is lost (to lowest order) and $\mathbf{E} \approx \mathbf{e}$, where $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$. Write down the equations of linear thermoelasticity (i.e. the linearized partial differential equations governing the displacement $\mathbf{u}$). To make the system complete, assume Fourier's law of heat conduction.
Derive an expression for the thermal strain tensor (a generalization of the coefficient of thermal expansion), giving the strain $\mathbf{e}$ in terms of the temperature as $\mathbf{e} = \boldsymbol{\alpha}(\theta - \theta_0)$ under zero stress.

3. Consider the one-dimensional free energy function ψ (at a fixed constant temperature) given by

$$\rho_0 \psi(e, \xi) = \tfrac{1}{2} E_1 (e - \xi)^2 + \tfrac{1}{2} E_2 \xi^2$$

together with the evolution law

$$\dot{\xi} = \alpha f.$$

[This system can be visualised as comprising a spring with modulus E_1, in series with a composite unit consisting of a spring with modulus E_2 in parallel with a dashpot with damping coeeficient $1/\alpha$; the internal variable ξ denotes the extension of this composite unit.]
Write the stress σ and the dissipative force f in terms of e and ξ.
(a) Eliminate f and ξ to obtain a relation between σ and e, and their derivatives with respect to t.
(b) Express $\xi(t)$ as a functional of $e(t)$ and deduce that

$$\sigma(t) = E_1 e(t) - \frac{E_1^2}{(E_1 + E_2)\tau} \int_{-\infty}^{t} \exp[-(t - t')/\tau] e(t')\, \mathrm{d}t',$$

where $\tau^{-1} = \alpha(E_1 + E_2)$. [This material is called a "standard linear solid".]

4. A given Green-elastic material is inextensible in the (initial) direction $\mathbf{N}$ (taken as a unit vector). Show that $\mathbf{N}^T\mathbf{F}^T\mathbf{F}\mathbf{N} = 1$. Show that the constitutive relation for such a material is $P_{Ii} = \partial W/\partial F_{iI} + q F_{iJ} N_J N_I$, where q is any scalar. Give a corresponding expression for Cauchy stress and interpret the "arbitrary" term in relation to the direction $\mathbf{n} = \mathbf{F}\mathbf{N}$. [*Look at* $n_j \sigma_{ji}$, *for instance.*]

5. In cylindrical polars (R, ϕ, Z), a thick tube occupies (initially) $R_1 < R < R_2$, $-\infty < Z < \infty$. A state of plane strain is induced by uniform pressure p on its inner surface so that, after deformation, $(R, \phi, Z) \to (r, \theta, z)$, where $z = Z$, $\theta = \phi$ and $r = f(R)$. Find the principal stretches and directions, and find $f(R)$ if the material is incompressible.

Deduce that the pressure $p(r_1)$ at inner radius r_1 is given by

$$p(r_1) = \int_{R_1}^{R_2} (W_{,2} - \lambda^{-2} W_{,1}) \lambda^{-1} \frac{\mathrm{d}R}{R},$$

where $\lambda = r/R$, and $W(\lambda_1, \lambda_2, \lambda_3)$ and its partial derivatives are evaluated at $(\lambda^{-1}, \lambda, 1)$.

6. A cube, which is constrained so that its edges remain aligned with the coordinate axes, is composed of (incompressible) neo-Hookean material, with energy function

$$W(\mathbf{F}) = \tfrac{1}{2}\mu(F_{iI}F_{iI} - 3).$$

It is subjected to dead-loading at its surface, to generate the uniform nominal stress $P_{11} = P_{22} = P_{33} = T$, $P_{Ii} = 0$ otherwise. Show that the deformation $\mathbf{F} = \mathbf{I}$ is always possible, but that seven distinct homogeneous deformations are possible if $T > (3/2^{2/3})\mu$. [*You will need to consider solutions of the equation* $\lambda^3 - (T/\mu)\lambda^2 + 1 = 0$ *or some equivalent.*]