3C7a **Perturbation and stability methods: Examples Sheet 1** Michaelmas 2006

A star $*$ denotes a difficult question, or part of a question, that should not be done at the expense of unstarred questions . You are welcome to use an algebraic manipulator if you think it would help. Corrections, suggestions and comments should be emailed to `j.m.rallison@damtp.cam.ac.uk`.

1. Find the appropriate scalings for the roots of

(*a*)
$$\epsilon x^4 - x^2 - x + 2 = 0\,,$$

(*b*)
$$\epsilon^2 x^3 + x^2 + 2x + \epsilon \;=\; 0\,.$$

Thence find two terms in the approximation for each root.

2. Find an asymptotic approximation to the exponential integral

$$E_n(x) \quad = \quad \int_1^\infty t^{-n} e^{-xt}\, dt\ ,$$

for real $n = \mathrm{ord}(1)$ and $x \to \infty$, and estimate the remainder. How big is the remainder for the best choice of the number of terms in the expansion?

3. Evaluate the first two terms as $r \to 0$, and the first two terms as $r \to \infty$, of

$$\int_0^\infty \frac{rx\, dx}{(r^2+x)^{3/2}(1+x)}\ .$$

$*$ Find the first four terms of the latter expansion (counting $\ln r$ and 1 as different orders of magnitude).

$*$4. The integral $I(\lambda)$ is defined by

$$I(\lambda) = \int_0^\infty s \exp\left(-s^2 + 2\lambda - \frac{\lambda^2}{s^2}\right) \mathrm{d}s\,.$$

Find the asymptotic expansion for $I(\lambda)$ as $\lambda \to 0$ correct to, and including, terms that are $\mathcal{O}(\lambda^2)$. It may help to recall that

$$\int_0^\infty \ln(t)\exp(-t)\,\mathrm{d}t = -\gamma\,,$$

where γ is Euler's constant.

$*$5. Let

$$\alpha_n = \int_0^1 \frac{1-(1-t)^n}{t}\, dt\ .$$

Show that as $n \to \infty$

$$\alpha_n \sim \ln n + \int_0^1 \frac{1-\exp(-t)}{t}\, dt - \int_1^\infty \frac{\exp(-t)}{t}\, dt\ .$$

Hence, or otherwise, deduce that

$$\int_0^\infty \ln(t)\exp(-t) dt = -\gamma\ ,$$

where γ is Euler's constant:

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n)\right) .$$

Thus find the first two terms of the asymptotic expansion as $\lambda \to \infty$ of

$$\lambda \int_0^1 \frac{\exp(-\lambda t)}{\ln\left(\frac{1}{2}t\right)}\, dt .$$

6. For real x and $x \to \infty$, find the full asymptotic behaviour of

$$K_0(x) \quad = \quad \int_1^\infty (t^2-1)^{-\frac{1}{2}} e^{-xt}\, dt .$$

7. For real x and $x \to \infty$, find the leading-order asymptotic behaviour of

   (*a*) $$\int_0^1 \sin(t)\, e^{-x\sinh^4 t}\, dt ,$$

   (*b*) $$\int_0^\infty e^{-xt-t^{-1}}\, dt .$$

8. For real n and $n \to \infty$, find the leading-order asymptotic behaviour of

$$J_n(n) \quad = \quad \frac{1}{\pi}\int_0^\pi \cos(n\sin t - nt)\, dt .$$

[Give your answer in a form that is explicitly real. It should involve $\Gamma(1/3)$.]

9. Find the asymptotic behaviour of

$$J_\nu(\nu\,\mathrm{sech}\,\alpha) \quad = \quad \frac{1}{2\pi\mathrm{i}}\int_{\infty-\mathrm{i}\pi}^{\infty+\mathrm{i}\pi} e^{\nu\,\mathrm{sech}\,\alpha\sinh t - \nu t}\, dt ,$$

for real ν and α, as $\nu \to \infty$ with first $\alpha > 0$ and second $\alpha = 0$. (There are lots of saddles here. A contour plot using MATLAB may help convince you that you have a steepest descent contour. The case $\alpha = 0$ has a cubic saddle where three ridges meet and $\phi'' = 0$.)

10. The function $f(y;\lambda)$ is defined by

$$f(y;\lambda) = \int_C \exp\left(\lambda(1+\mathrm{i}y)z - \tfrac{1}{3}z^3\right) dz,$$

where y and λ are real, and the contour C starts from $z = 0$ and extends to $z = \infty$ in the sector $|\arg(z)| < \pi/6$.

   (a) Find the leading-order asymptotic behaviour of $f(0;\lambda)$ as $\lambda \to -\infty$.
   (b) Find the leading-order asymptotic behaviour of $f(0;\lambda)$ as $\lambda \to +\infty$.
   (c) By considering the solutions deduced in parts (*a*) and (*b*), and the steepest descent contours, find the leading-order asymptotic behaviour of f for $0 \le y < \infty$. In particular:
      i. state clearly your choice of integration contour;
      ii. for $\lambda \gg 1$ comment on how the asymptotic behaviour of the solution differs according as $0 \le y < y_c$ and $y_c < y < \infty$, where y_c should be identified.

(d) Show that $f(y;\lambda)$ satisfies the differential equation

$$f_{yy} + \lambda^3(1+\mathrm{i}y)f = -\lambda^2 \ ,$$

with boundary conditions $f \to 0$ as $|y| \to \infty$.

(e) * In relation to this equation, why is it that

$$f = -\frac{1}{\lambda(1+\mathrm{i}y)} - \frac{2}{\lambda^4(1+\mathrm{i}y)^4} + \dots \ ,$$

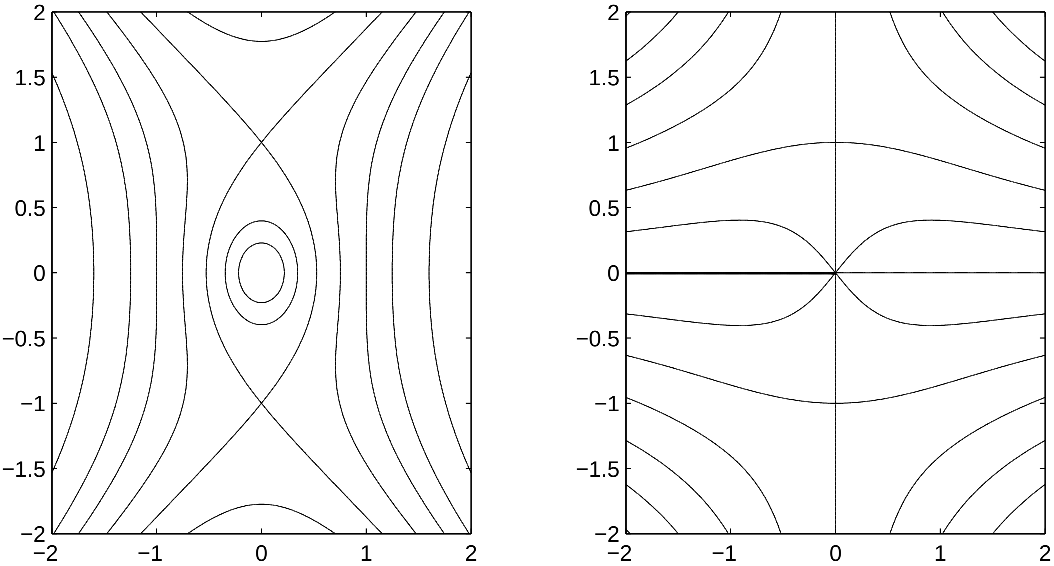
is *not* always a uniformly valid asymptotic approximation for $|\lambda| \gg 1$? * [This issue will be considered in the matched asymptotic expansions section of the course later.]

11. The integral $\mathcal{E}_n(z)$ is defined by

$$\mathcal{E}_n(z) = \int_z^\infty \frac{\mathrm{e}^{-t^2}}{t^{2n}}\,\mathrm{d}t\,,$$

where z is a complex number ($-\pi < \arg z \leq \pi$), n is an integer, and the integration contour starts from $t = z$ and extends to $t = \infty$ in the sector $|\arg(t)| < \pi/4$.

(a) For $z = n^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\theta}$, and for all values of θ in the range $-\pi < \theta \leq \pi$, obtain the appropriate leading-order asymptotic expansion of $\mathcal{E}_n(z)$ as $n \to \infty$. *Hints:* (i) see the figure below, and (ii) while you should find that the *form* of leading-order asymptotic expansion changes at certain values of θ, you should *not* examine in detail the nature of the change at these 'transition' values of θ.



Contours of constant $\mathrm{Re}(w^2 + 2\log(w))$ and $\mathrm{Im}(w^2 + 2\log(w))$ for complex w.
The branch cut is taken down the negative real axis.

(b) It is possible to show by repeated integration by parts (do *not* prove this) that

$$\tfrac{1}{2}\pi^{1/2}(1-\mathrm{erf}(z)) \equiv \mathcal{E}_0(z) = \frac{\mathrm{e}^{-z^2}}{2z}\left(\sum_{r=0}^{n-1}\frac{(-)^r(2r)!}{(2z)^{2r}r!}\right) + \mathcal{R}_n(z)\,,$$

where the remainder $\mathcal{R}_n(z)$ is given by

$$\mathcal{R}_n(z) = \frac{(-)^n(2n)!}{2^{2n}n!}\,\mathcal{E}_n(z)\,.$$

An 'optimal' asymptotic series for $\mathcal{E}_0(z)$ as $|z| \to \infty$ can be formed by truncating the series at the smallest term. For a given value of z, for what approximate value of n should the series be truncated? For this 'optimal' truncation, estimate the size of the remainder. Briefly discuss the dependence of the remainder on $\arg z$ $(-\pi < \arg z \leq \pi)$.

Stirling's asymptotic approximation for the Gamma function may prove useful:

$$(n-1)! = \Gamma(n) \sim \left(\frac{2\pi}{n}\right)^{\frac{1}{2}} n^n\,\mathrm{e}^{-n}\ .$$

$*$ 12. (a) Let

$$f(\lambda) = \int_{\mathcal{C}} \frac{G(z)}{z-z_0}\exp(-\mathrm{i}\lambda W(z))\,dz\ ,$$

where G and W are analytic near the contour $\mathcal{C}$ and z_0, $W(z)$ has a single saddle point at $z = z_0$, and $\mathcal{C}$ passes from one 'valley' of $\Im(W)$ with respect to z_0 to another, avoiding z_0 in a clockwise manner. Show that

$$f(\lambda) \sim -\mathrm{i}\pi G(z_0)\exp(-\mathrm{i}\lambda W(z_0))\ .$$

(b) The function $g(\theta;\lambda)$ is defined by

$$g(\theta;\lambda) = \int_\Gamma \frac{1}{(z-z_0)(z^2-1)^{\frac{1}{2}}}\exp\left(\mathrm{i}\lambda\left((z^2-1)^{\frac{1}{2}}\theta - z\right)\right)\,dz\ ,$$

where θ, z_0 and λ are real and positive, and the contour Γ goes from $-\infty$ to ∞ passing above the three singularities of the integrand. Take the branch cuts for $(z^2-1)^{\frac{1}{2}}$ to be lines drawn towards $\Im(z) = -\infty$ from the points $z = \pm 1$. Obtain the leading-order asymptotic behaviour of $g(\theta;\lambda)$ as $\lambda \to \infty$ on the assumption that $0 < \theta < 1$ and $z_0 > 1$; be careful to discuss all cases.