

3C7a

## Perturbation Methods: Examples Sheet 1

Michaelmas 2018

A star \* denotes a question, or part of a question, that should not be done at the expense of unstarred questions. Corrections, suggestions and comments should be emailed to [S.J.Cowley@maths.cam.ac.uk](mailto:S.J.Cowley@maths.cam.ac.uk).

If you would like questions 3, 6 and 8 marked in advance of the first Examples Class on 5 November, please note the following:

- the deadline for handing in your work is noon on Friday 2 November;
- please place your work in the folder in Stephen Cowley's DAMTP pigeonhole in the CMS;
- please put your full name and CRSid on your work.

1. Find the appropriate scalings for each root of

$$(a) \quad \epsilon x^4 - x^2 - x + 2 = 0,$$

$$(b) \quad \epsilon^2 x^3 + x^2 + 2x + \epsilon = 0.$$

Hence find two terms in the approximation for each root.

2. Find an asymptotic approximation to the exponential integral

$$E_n(x) = \int_1^\infty t^{-n} e^{-xt} dt,$$

for real  $n = \text{ord}(1)$  and  $x \rightarrow \infty$ , and estimate the remainder. How big is the remainder for the best choice of the number of terms in the expansion?

3. Evaluate the first two terms as  $r \rightarrow 0$ , and the first two terms as  $r \rightarrow \infty$ , of

$$\int_0^\infty \frac{rx dx}{(r^2 + x)^{3/2}(1 + x)}.$$

\* 4. The integral  $I(\lambda)$  is defined by

$$I(\lambda) = \int_0^\infty s \exp\left(-s^2 + 2\lambda - \frac{\lambda^2}{s^2}\right) ds.$$

Find the asymptotic expansion for  $I(\lambda)$  as  $\lambda \rightarrow 0$  correct to, and including, terms that are  $\mathcal{O}(\lambda^2)$ . It may help to recall that

$$\int_0^\infty \ln(t) \exp(-t) dt = -\gamma,$$

where  $\gamma$  is Euler's constant.

5. For real  $x$  and  $x \rightarrow \infty$ , find the full asymptotic behaviour of

$$K_0(x) = \int_1^\infty (t^2 - 1)^{-\frac{1}{2}} e^{-xt} dt.$$

6. For real  $x$  and  $x \rightarrow \infty$ , find the leading-order asymptotic behaviour of

$$(a) \quad \int_0^1 \sin(t) e^{-x \sinh^4 t} dt,$$

$$(b) \quad \int_0^\infty e^{-xt - t^{-1}} dt.$$

7. For real  $n$  and  $n \rightarrow \infty$ , find the leading-order asymptotic behaviour of

$$J_n(n) = \frac{1}{\pi} \int_0^\pi \cos(n \sin t - nt) dt .$$

[Give your answer in a form that is explicitly real. It should involve  $\Gamma(1/3)$ .]

8. Find the asymptotic behaviour of

$$J_\nu(\nu \operatorname{sech} \alpha) = \frac{1}{2\pi i} \int_{\infty - i\pi}^{\infty + i\pi} e^{\nu \operatorname{sech} \alpha \sinh t - \nu t} dt ,$$

for real  $\nu$  and  $\alpha$ , as  $\nu \rightarrow \infty$  with first  $\alpha > 0$  and second  $\alpha = 0$ . (There are lots of saddles here. A contour plot using MATLAB may help convince you that you have a steepest descent contour. The case  $\alpha = 0$  has a cubic saddle where three ridges meet and  $\phi'' = 0$ .)

9. The function  $f(y; \lambda)$  is defined by

$$f(y; \lambda) = \int_C \exp(\lambda(1 + iy)z - \frac{1}{3}z^3) dz,$$

where  $y$  and  $\lambda$  are real, and the contour  $C$  starts from  $z = 0$  and extends to  $z = \infty$  in the sector  $|\arg(z)| < \pi/6$ .

- (a) Find the leading-order asymptotic behaviour of  $f(0; \lambda)$  as  $\lambda \rightarrow -\infty$ .
- (b) Find the leading-order asymptotic behaviour of  $f(0; \lambda)$  as  $\lambda \rightarrow +\infty$ .
- (c) By considering the solutions deduced in parts (a) and (b), and the steepest descent contours, find the leading-order asymptotic behaviour of  $f$  for  $0 \leq y < \infty$ . In particular:
  - i. state clearly your choice of integration contour;
  - ii. for  $\lambda \gg 1$  comment on how the asymptotic behaviour of the solution differs according as  $0 \leq y < y_c$  and  $y_c < y < \infty$ , where  $y_c$  should be identified.
- (d) Show that  $f(y; \lambda)$  satisfies the differential equation

$$f_{yy} + \lambda^3(1 + iy)f = -\lambda^2 ,$$

with boundary conditions  $f \rightarrow 0$  as  $|y| \rightarrow \infty$ .

- (e) \* In relation to this equation, why is it that

$$f = -\frac{1}{\lambda(1 + iy)} - \frac{2}{\lambda^4(1 + iy)^4} + \dots ,$$

is *not* always a uniformly valid asymptotic approximation for  $|\lambda| \gg 1$ ? \* [This issue will be considered in the matched asymptotic expansions section of the course later.]