

3C7a

Perturbation Methods: Examples Sheet 1

Michaelmas 2017

A star * denotes a question, or part of a question, that should not be done at the expense of unstarred questions. Corrections, suggestions and comments should be emailed to L.J.Ayton@maths.cam.ac.uk.

If you would like questions 3, 6 and 8 marked in advance of the first Examples Class on 6 November, please note the following:

- the deadline for handing in your work is noon on Friday 3 November;
- please place your work in the folder in Stephen Cowley's DAMTP pigeonhole in the CMS;
- please put your name and CRSid on your work.

1. Find the appropriate scalings for each root of

$$(a) \quad \epsilon x^4 - x^2 - x + 2 = 0,$$

$$(b) \quad \epsilon^2 x^3 + x^2 + 2x + \epsilon = 0.$$

Hence find two terms in the approximation for each root.

2. Find an asymptotic approximation to the exponential integral

$$E_n(x) = \int_1^\infty t^{-n} e^{-xt} dt,$$

for real $n = \text{ord}(1)$ and $x \rightarrow \infty$, and estimate the remainder. How big is the remainder for the best choice of the number of terms in the expansion?

3. Evaluate the first two terms as $r \rightarrow 0$, and the first two terms as $r \rightarrow \infty$, of

$$\int_0^\infty \frac{rx dx}{(r^2 + x)^{3/2}(1 + x)}.$$

*4. The integral $I(\lambda)$ is defined by

$$I(\lambda) = \int_0^\infty s \exp\left(-s^2 + 2\lambda - \frac{\lambda^2}{s^2}\right) ds.$$

Find the asymptotic expansion for $I(\lambda)$ as $\lambda \rightarrow 0$ correct to, and including, terms that are $\mathcal{O}(\lambda^2)$. It may help to recall that

$$\int_0^\infty \ln(t) \exp(-t) dt = -\gamma,$$

where γ is Euler's constant.

5. For real x and $x \rightarrow \infty$, find the full asymptotic behaviour of

$$K_0(x) = \int_1^\infty (t^2 - 1)^{-\frac{1}{2}} e^{-xt} dt.$$

6. For real x and $x \rightarrow \infty$, find the leading-order asymptotic behaviour of

$$(a) \quad \int_0^1 \sin(t) e^{-x \sinh^4 t} dt,$$

$$(b) \quad \int_0^\infty e^{-xt - t^{-1}} dt.$$

7. For real n and $n \rightarrow \infty$, find the leading-order asymptotic behaviour of

$$J_n(n) = \frac{1}{\pi} \int_0^\pi \cos(n \sin t - nt) dt .$$

[Give your answer in a form that is explicitly real. It should involve $\Gamma(1/3)$.]

8. Find the asymptotic behaviour of

$$J_\nu(\nu \operatorname{sech} \alpha) = \frac{1}{2\pi i} \int_{\infty - i\pi}^{\infty + i\pi} e^{\nu \operatorname{sech} \alpha \sinh t - \nu t} dt ,$$

for real ν and α , as $\nu \rightarrow \infty$ with first $\alpha > 0$ and second $\alpha = 0$. (There are lots of saddles here. A contour plot using MATLAB may help convince you that you have a steepest descent contour. The case $\alpha = 0$ has a cubic saddle where three ridges meet and $\phi'' = 0$.)

9. The function $f(y; \lambda)$ is defined by

$$f(y; \lambda) = \int_C \exp(\lambda(1 + iy)z - \frac{1}{3}z^3) dz,$$

where y and λ are real, and the contour C starts from $z = 0$ and extends to $z = \infty$ in the sector $|\arg(z)| < \pi/6$.

- Find the leading-order asymptotic behaviour of $f(0; \lambda)$ as $\lambda \rightarrow -\infty$.
- Find the leading-order asymptotic behaviour of $f(0; \lambda)$ as $\lambda \rightarrow +\infty$.
- By considering the solutions deduced in parts (a) and (b), and the steepest descent contours, find the leading-order asymptotic behaviour of f for $0 \leq y < \infty$. In particular:
 - state clearly your choice of integration contour;
 - for $\lambda \gg 1$ comment on how the asymptotic behaviour of the solution differs according as $0 \leq y < y_c$ and $y_c < y < \infty$, where y_c should be identified.
- Show that $f(y; \lambda)$ satisfies the differential equation

$$f_{yy} + \lambda^3(1 + iy)f = -\lambda^2 ,$$

with boundary conditions $f \rightarrow 0$ as $|y| \rightarrow \infty$.

- (e) * In relation to this equation, why is it that

$$f = -\frac{1}{\lambda(1 + iy)} - \frac{2}{\lambda^4(1 + iy)^4} + \dots ,$$

is *not* always a uniformly valid asymptotic approximation for $|\lambda| \gg 1$? * [This issue will be considered in the matched asymptotic expansions section of the course later.]