

**Example Sheet 2: Multiple scales**

1. Apply the method of multiple scales to the Duffing equation

$$\frac{d^2u}{dt^2} + u + \epsilon u^3 = 0 \quad \text{with} \quad \epsilon \ll 1$$

with initial conditions  $du/dt = 0$ ,  $u = 1$  at  $t = 0$  to find the long-time evolution uniformly in  $\epsilon t \leq O(1)$ . Repeat your analysis when the cubic term in the Duffing equation is replaced by  $\epsilon(du/dt)^3$ .

2. Consider the equation

$$\frac{d^2u}{dt^2} + u - \epsilon u^2 = 0 \quad \text{with} \quad \epsilon \ll 1.$$

Find the order in  $\epsilon$  at which a secular term may first arise, and hence use multiple scales to determine the long-time behaviour of the leading-order solution.

**3. Parametric resonance**

The Mathieu equation can be written

$$\frac{d^2u}{dt^2} + (1 + \epsilon^2 f + \epsilon \cos t)u = 0 \quad ,$$

where  $f$  is a fixed constant and  $\epsilon \ll 1$ . Find the range of values of  $f$  for which the solution to this equation grows in time.

**4. Stroboscopic method**

A perturbed oscillator satisfies

$$\frac{d^2u}{dt^2} + u = \epsilon f\left(\frac{du}{dt}, u, t\right) \quad .$$

Use the method of multiple scales to show that the leading-order solution takes the form  $u = R(T) \cos(t + \phi(T))$ , where  $T = \epsilon t$ ,

$$\frac{dR}{dT} = - \langle f \sin(t + \phi) \rangle \quad R \frac{d\phi}{dT} = - \langle f \cos(t + \phi) \rangle \quad ,$$

and  $\langle \dots \rangle$  denotes the average over the fast time period  $0 < t < 2\pi$ . Use these results to re-derive your answers to question 1.

5. Find the leading-order solution, valid for  $T = \epsilon t = O(1)$ , to the system

$$\begin{aligned} \frac{d^2x}{dt^2} + \epsilon y \frac{dx}{dt} + x &= y^2 \\ \frac{dy}{dt} &= \epsilon(1 + x - y - y^2), \end{aligned}$$

with  $x = 1$ ,  $dx/dt = 0$ ,  $y = 0$  when  $t = 0$ .

**6. Schrodinger's equation for a potential well.**

Find the *large* energy eigenvalues,  $E$ , for the equation

$$\frac{d^2\psi}{dx^2} + (E - x^2)\psi = 0 \quad \psi \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty.$$

How do the eigenstates behave for  $|x| \rightarrow \infty$ ?

7. Determine the leading-order solution, which is uniformly valid in  $x$ , of

$$\epsilon^2 \frac{d^2y}{dx^2} = (1 + x^2)^2 y$$

with  $y = A$ ,  $dy/dx = B$  at  $x = 0$ .

**8. Waves in a cylindrical duct.**

Consider a straight circular duct of radius 1 and sound waves of frequency  $k_0$ . Show that the solutions of  $\nabla^2\phi + k_0^2\phi = 0$  which satisfy  $\partial\phi/\partial r = 0$  on  $r = 1$  take the form

$$J_m(j'_{ms}r) \exp(ikx + im\theta - i\omega t) \quad s = 1, 2, 3, \dots,$$

where  $m, s$  are integers,  $k^2 = k_0^2 - (j'_{ms})^2$ , and  $j'_{ms}$  is the  $s$ th root of the equation  $J'_m(z) = 0$  on the real axis.

[The Bessel function  $y = J_m(z)$  is the solution of the equation

$$z(zy')' + (z^2 - m^2)y = 0$$

that is regular at  $z = 0$ .]

9. Using your answer to question 8, describe the propagation of a single acoustic mode along a circular pipe whose radius,  $R(X)$ , varies slowly. You may find useful the orthogonality relation

$$\int_0^1 J_m(j_{ms}r) J_m(j_{mt}r) r dr = 0 \quad \text{when} \quad s \neq t.$$

[No need to consider the possible occurrence of a turning point.]

10. Consider a three-dimensional medium with sound speed  $c_0 = c_0(Z)$ , i.e., depending on only one coordinate direction. Use the ray tracing equation to derive Snell's Law, i.e. that if points on the ray path are  $(X(Z), Y(Z), Z)$  then

$$\frac{X'}{c_0 \sqrt{1 + X'^2 + Y'^2}} \quad \text{and} \quad \frac{Y'}{c_0 \sqrt{1 + X'^2 + Y'^2}}$$

are constants of the motion. Without detailed calculation, explain what might happen to sound waves propagating in an ocean where the sound speed decreases linearly with depth down to some critical depth  $Z = h$ , before increasing linearly with depth below  $Z = h$ .

### 11. Ray theory in low-Mach number shear flow

Consider a flow with mean velocity  $\mathbf{U} = (U(\epsilon y), 0, 0)$ , but with uniform mean density  $\rho_0$  and sound speed  $c_0$ . The equations describing the propagation of sound waves with density  $\rho'$  and velocity  $\mathbf{u}'$  are

$$\frac{\partial \rho'}{\partial t} + \mathbf{U} \cdot \nabla \rho' + \rho_0 \nabla \cdot \mathbf{u}' = 0$$

and

$$\rho_0 \left( \frac{\partial \mathbf{u}'}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{U} \right) = -\nabla (c_0^2 \rho') .$$

By writing the fluctuating quantities in the form

$$A(\mathbf{X}) \exp(-i\omega t + i\theta(\mathbf{X})/\epsilon) ,$$

where  $\mathbf{X} = \epsilon \mathbf{x}$ , find the Eikonal equations. What will happen to sound propagating over ground through a typical wind-driven boundary layer?

[No need to derive the Transport equation here.]

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