3C7c **Perturbation Methods: Examples Sheet 3** Michaelmas 2017

A star $*$ denotes a question, or part of a question, that should not be done at the expense of unstarred questions (e.g. you might like to miss them out first time through the sheet). You are welcome to use an algebraic manipulator if you think it would help. Corrections, suggestions and comments should be emailed to `S.J.Cowley@maths.cam.ac.uk`.

If you would like questions 2, 4, 7 and 9 marked in advance of the third Examples Class on 22 January 2018, please note the following:

- *the deadline for handing in your work is noon on Friday 19 January 2018;*
- *please place your work in the folder in Stephen Cowley's DAMTP pigeonhole in the CMS;*
- *please put your full name and CRSid on your work.*

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1. Apply the method of multiple scales to the Duffing equation

$$\frac{d^2u}{dt^2} + u + \epsilon u^3 = 0 \quad \text{with} \quad \epsilon \ll 1$$

with initial conditions $du/dt = 0$, $u = 1$ at $t = 0$ to find the long-time evolution uniformly in $\epsilon t \leqslant O(1)$. Repeat your analysis when the cubic term in the Duffing equation is replaced by $\epsilon(du/dt)^3$.

2. *Stroboscopic method*. A perturbed oscillator satisfies

$$\frac{d^2u}{dt^2} + u = \epsilon f\left(\frac{du}{dt}, u, t\right) \quad .$$

By using the method of multiple scales show that the leading-order solution takes the form $u = R(T)\cos(t + \phi(T))$, where $T = \epsilon t$,

$$\frac{dR}{dT} = - < f\sin(t+\phi) > \qquad R\frac{d\phi}{dT} = - < f\cos(t+\phi) > \quad ,$$

and $< .. >$ denotes the average over the fast time period $0 < t < 2\pi$. Use these results to re-derive your answers to question 1.

3. Find the leading-order approximation to the general solution for $x(t;\epsilon)$ and $y(t;\epsilon)$ satisfying

$$\begin{aligned} \frac{d^2x}{dt^2} + 2\epsilon y\frac{dx}{dt} + x &= 0 \ , \\ \frac{dy}{dt} &= \tfrac{1}{2}\epsilon \ln x^2 \ , \end{aligned}$$

which is valid for $t = \mathrm{ord}(1/\epsilon)$ as $\epsilon \to 0$. You may quote the result

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\cos^2\theta \, d\theta = -\ln 4 \ .$$

4. Solve the Mathieu equation

$$\ddot{y} + (\omega^2 + \epsilon\cos t)y = 0 \ ,$$

for the case when $\omega = \frac{1}{2} + \epsilon\omega_1 + \ldots$. Identify the stability boundary correct to $\mathrm{ord}(\epsilon)$.

*Explain why it is necessary to introduce a slow time $\mathcal{T} = \epsilon^{\frac{3}{2}}t$ in order to calculate the stability boundary correct to $\mathrm{ord}(\epsilon^2)$ and perform the calculation.

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5. The function $u(t;\epsilon)$ satisfies the governing equation

$$\frac{d^2u}{dt^2} - \lambda\epsilon^2 t\frac{du}{dt} + u = \epsilon\gamma u^2\frac{du}{dt} ,$$

and the initial conditions

$$u = 2a , \quad \text{and} \quad \frac{du}{dt} = 0 \quad \text{at} \quad t = 0 ,$$

where $0 < \epsilon \ll 1$, and λ, γ and a are order one constants. By ascertaining at what order of ϵ a secularity first appears in the regular perturbation expansion for $u(t;\epsilon)$, or otherwise, find a solution for $|u(t)|^2$ that is uniformly valid for large times. If $\lambda > 0$, sketch typical solutions for $|u|^2$ for both $\gamma > 0$ and $\gamma < 0$. Sketch the squared amplitude as a function of time for different values of a, with special emphasis on the case $|a| \ll 1$.

6. Find the leading-order approximation which is valid for times $t = \mathrm{ord}(\epsilon^{-1})$ as $\epsilon \to 0$, to the solution $x(t;\epsilon)$ and $y(t;\epsilon)$ satisfying

$$\begin{aligned} \frac{dx}{dt} + x^2 y\cos t &= \epsilon(x - 2x^2) , \\ \frac{dy}{dt} &= \epsilon\left(1 - \frac{\sin t}{x}\right) , \end{aligned}$$

with $x = 1$ and $y = 0$ at $t = 0$.

7. Use the transformation

$$x(t,\epsilon) = \Re\left[r(\epsilon t,\epsilon)\exp\left(\mathrm{i}\int^t \sigma(\epsilon q,\epsilon)\,dq\right)\right] ,$$

to obtain a higher order approximation correct to $O(\epsilon^2)$ to the equation

$$\ddot{x} + f(\epsilon t)x = 0 ,$$

where the function f is real.

$*$ 8. Find the large eigenvalue solutions of the equation

$$y'' + \lambda(1 - x^2)^2 y = 0 ,$$

subject to $y = 0$ at $x = \pm 1$. At the ends $x = \pm 1$ you will need to use turning point solutions like

$$(1 - x^2)^{1/2}\mathrm{J}_{1/4}(\lambda^{1/2}(1 - x^2)^2/4),$$

and then use

$$\mathrm{J}_{1/4}(z) \sim (2/\pi z)^{1/2}\cos(z - 3\pi/8) \quad \text{as} \quad z \to \infty.$$

9. Sound waves propagating through a slow-varying mean flow satisfy the equations

$$\rho_0(\tilde{u}_t + (U\tilde{u})_z) = -c_0^2\tilde{\rho}_z , \quad \tilde{\rho}_t + (U\tilde{\rho})_z = -\rho_0\tilde{u}_z ,$$

where the wavespeed c_0 and the undisturbed density ρ_0 are constants, $\tilde{u}(z,t)$ and $\tilde{\rho}(z,t)$ are the perturbation velocity and density respectively, and $U(\epsilon z,\epsilon t)$ is the slowly-varying mean flow. By seeking solutions of the form,

$$(\tilde{\rho},\tilde{u}) = ((A_0,B_0)(\epsilon z,\epsilon t) + \epsilon(A_1,B_1)(\epsilon z,\epsilon t) + \ldots)\exp(\mathrm{i}\theta(\epsilon z,\epsilon t)/\epsilon) + \text{c.c.} ,$$

show that the wave action E_r/ω_r is conserved, where

$$E_r = \frac{c_0^2|A_0|^2}{2\rho_0} , \quad \text{and} \quad \omega_r = \omega - kU .$$

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Some Extra MAE Questions

10. *2016, Paper 336, Q2.*

(a) The function $y(x)$ satisfies the differential equation

$$\varepsilon y'' + 2xy' - 2xy = 0\,,$$

and boundary conditions

$$y(0) = 0\,, \quad y(1) = e\,,$$

where $0 < \varepsilon \ll 1$. By means of matched asymptotic expansions find the solution for $y(x)$ correct to and including $O(\varepsilon)$ terms for $0 \leqslant x \leqslant 1$.

Hints.

(i) Recall that

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \quad \text{and} \quad \operatorname{erf}(\infty) = 1\,.$$

(ii) A particular solution for $Y(z)$ to

$$Y'' + 2zY' = 2a_1 z \operatorname{erf}(z) + 2a_2 z + a_3 z e^{-z^2}\,,$$

where a_1, a_2 and a_3 are constants, is

$$Y = a_1 \left(\tfrac{2}{\sqrt{\pi}} e^{-z^2} + z\operatorname{erf}(z) \right) + a_2 z - \tfrac{1}{4} a_3 z e^{-z^2}\,.$$

(iii) A particular solution for $Y(z)$ to

$$Y'' + 2zY' = 2z^2 \operatorname{erf}(z)$$

is

$$Y = \tfrac{1}{2} z^2 \operatorname{erf}(z) + \tfrac{1}{\sqrt{\pi}} z e^{-z^2} - \textstyle\int_0^z e^{-t^2} \int_0^t e^{u^2} \operatorname{erf}(u) du dt\,,$$

where

$$\int_0^z e^{-t^2} \int_0^t e^{u^2} \operatorname{erf}(u) du dt \to \tfrac{1}{2} \log z + C \quad \text{as} \quad z \to \infty\,,$$

and C is to be taken as a known constant.

(b) The function $z(x)$ satisfies the differential equation

$$\varepsilon x z'' + z' + 2xz = 0\,,$$

and boundary conditions

$$z(\varepsilon) = 0\,, \quad z(1) = e^{-1}\,,$$

where $0 < \varepsilon \ll 1$. By means of matched asymptotic expansions, find the solution for $z(x)$ correct to and including $O(\varepsilon)$ terms, in both inner and outer regions for $\varepsilon \leqslant x \leqslant 1$.

Hints. A particular solution to

$$z' + 2xz = 2x(1 - 2x^2)e^{-x^2} \qquad \text{is} \qquad z = x^2(1 - x^2)e^{-x^2}\,,$$

and a particular solution to

$$z'' + z' = 2xe^{-x} \qquad \text{is} \qquad z = -x(2 + x)e^{-x}\,.$$

11. *2004, Paper 76, Q3.* The function $y(x)$ satisfies the differential equation

$$\varepsilon\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + xy + y^2 = 0\,.$$

If $y(a) = \alpha$ and $y(1) = \beta$, with $0 < a < 1$, identify for what values of α and β solutions can be found in the limit $\varepsilon \to 0$ on the assumption that there are no rapid oscillations or 'internal' boundary layers away from the end points. Sketch your solution[s] indicating whether there is a unique solution. Briefly discuss whether additional solutions with internal boundary layers can be *easily* ruled out.

Next suppose that $a = \alpha = 0$, i.e. $y(0) = 0$. Derive the governing equation in the inner region near $x = 0$ and state the boundary conditions a solution should satisfy. *Without solving the equation exactly*, discuss whether a solution satisfying the boundary conditions is likely to exist, e.g. on the basis of linearising the equation in the 'intermediate matching region' and discussing its solutions.

Comment. You may quote the result that

$$\int \frac{\mathrm{d}y}{\sqrt{a^3 - 3ay^2 - 2y^3}} = -\frac{2}{\sqrt{3a}}\,\mathrm{arctanh}\left(\sqrt{\frac{a-2y}{3a}}\right)\,.$$

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