3C7c **PSM: Examples Sheet 3** Michaelmas 2008

A star $*$ denotes a question, or part of a question, that should not be done at the expense of unstarred questions (e.g. you might like to miss them out first time through the sheet). You are welcome to use an algebraic manipulator if you think it would help. Corrections, suggestions and comments should be emailed to `S.J.Cowley@damtp.cam.ac.uk`.

1. The function $y(x;\epsilon)$ satisfies

$$\epsilon y'' + (1+\epsilon)y' + y \;=\; 0 \quad \text{in} \quad 0 \leqslant x \leqslant 1 \;,$$

and is subject to boundary conditions $y=0$ at $x=0$ and $y=e^{-1}$ at $x=1$. Find two terms in the outer approximation, applying only the boundary condition at $x=1$. Next find two terms in the inner approximation for the $\text{ord}(\epsilon)$ boundary layer near to $x=0$; apply only the boundary condition at $x=0$. Finally determine the constants of integration in the inner approximation by matching.

2. The function $y(x;\epsilon)$ satisfies

$$\epsilon y'' + x^{1/2}y' + y \;=\; 0 \quad \text{in} \quad 0 \leqslant x \leqslant 1 \;,$$

and is subject to the boundary conditions $y=0$ at $x=0$ and $y=1$ at $x=1$. First find the rescaling for the boundary layer near $x=0$, and obtain the leading order inner approximation. Then find the leading order outer approximation and match the two approximations.

3. The function $y(x;\epsilon)$ satisfies

$$(x+\epsilon y)y' + y = 1 \quad \text{with} \quad y(1) = 2 \;.$$

Find $y(0)$ correct to $\text{ord}(1)$.

4. The function $y(x,\epsilon)$ satisfies

$$\epsilon y'' + yy' - y \;=\; 0 \quad \text{in} \quad 0 \leqslant x \leqslant 1 \;,$$

and is subject to the boundary condition $y=0$ at $x=0$ and $y=3$ at $x=1$. Assuming that there is a boundary layer only near $x=0$, find the leading order terms in the outer and inner approximations and match them.

Suppose now the boundary conditions are replaced by $y=-\frac{3}{4}$ at $x=0$ and $y=\frac{5}{4}$ at $x=1$. Show that the boundary layer moves to an intermediate position which is determined by the property of the inner solution that y jumps within the boundary layer from $-M$ to M, for some value M. Find the leading order matched asymptotic expansions.

5. Consider the following problem which has an outer, an inner and an inner-inner inside the inner

$$x^3 y' \;=\; \epsilon\left((1+\epsilon)x + 2\epsilon^2\right)y^2 \quad \text{in} \quad 0 < x < 1 \;,$$

with $y(1) = 1-\epsilon$. Calculate two terms of the outer, then two of the inner, and finally one for the inner-inner. At each stage find the rescaling required for the next layer by examining the non-uniformity of the asymptoticness in the current layer.

6. The function $y(x;\epsilon)$ satisfies

$$(\epsilon + x)y' = \epsilon y \quad \text{with} \quad y(1) = 1 \;.$$

Find $y(0)$ correct to $\text{ord}(\epsilon^2)$.

7. The function $f(r,\epsilon)$ satisfies the equation

$$f_{rr} + \tfrac{2}{r} f_r + \frac{1}{2}\epsilon^2(1-f^2) \;=\; 0 \quad \text{in} \quad r>1\,,$$

and is subject to the boundary conditions

$$f=0 \quad \text{at} \quad r=1 \quad \text{and} \quad f\to 1 \quad \text{as} \quad r\to\infty\,.$$

Using the asymptotic sequence 1, ϵ, $\epsilon^2 \ln \frac{1}{\epsilon}$, ϵ^2, obtain asymptotic expansions for f at fixed r as $\epsilon \searrow 0$ and at fixed $\rho = \epsilon r$ as $\epsilon \searrow 0$. Match the expansions using the intermediate variable $\eta = \epsilon^\alpha r$ with $0<\alpha<1$.

Hint. You may quote that the solution to the equation

$$y_{xx} + \tfrac{2}{x} y_x - y \;=\; \frac{e^{-2x}}{x^2}\,,$$

subject to the condition $y\to 0$ as $x\to\infty$, is

$$y \;=\; A\frac{e^{-x}}{x} \;+\; \frac{1}{2x}\int_x^\infty \frac{e^{-x-t}-e^{x-3t}}{t}\,dt\,,$$

with A a constant. Further as $x\to 0$

$$y \;\sim\; \frac{2A+\ln 3}{2x} \;+\; \ln x - A + \gamma + \frac{1}{2}\ln 3 - 1\,.$$

8. The function $y(x)$ satisfies the equation

$$\varepsilon\frac{d^2y}{dx^2} + \left(1+\frac{2\varepsilon}{x}-\frac{2\varepsilon^3}{x^2}\right)\frac{dy}{dx} + \frac{2y}{x} = 0\,,$$

where $\varepsilon>0$, together with the boundary conditions

$$y(0)=\gamma \quad \text{and} \quad y(1)=\varepsilon^3\,,$$

where γ is a constant.

If $\varepsilon \ll 1$ find the order one value of γ for which an asymptotic solution can be found such that $y(x)$ is no larger than order one for $0\leqslant x\leqslant 1$.

Briefly comment on whether the problem as posed specifies a unique solution.

Hints. Note that

$$y_{xx} + \left(1+\frac{2}{x}\right)y_x + \frac{2y}{x} = \frac{1}{x^2}\left(x^2 y_x + x^2 y\right)_x\,,$$

and that the general solution to

$$y_{xx} + \left(\frac{2}{x}-\frac{2}{x^2}\right)y_x = 0\,,$$

is

$$y = A\exp\left(-\frac{2}{x}\right) + B\,,$$

where A and B are constants.

$*$9. The function $y(x)$ satisfies the differential equation

$$\varepsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + xy + y^2 = 0\,.$$

If $y(a) = \alpha$ and $y(1) = \beta$, with $0 < a < 1$, identify for what values of α and β solutions can be found in the limit $\varepsilon \to 0$ on the assumption that there are no rapid oscillations or 'internal' boundary layers away from the end points. Sketch your solution[s] indicating whether there is a unique solution. Briefly discuss whether additional solutions with internal boundary layers can be easily ruled out.

Next suppose that $a = \alpha = 0$, i.e. $y(0) = 0$. Derive the governing equation in the inner region near $x = 0$ and state the boundary conditions a solution should satisfy. Without solving the equation exactly, discuss whether a solution satisfying the boundary conditions is likely to exist, e.g. on the basis of linearising the equation in the 'intermediate matching region' and discussing its solutions.

Comment. You may quote the result that

$$\int \frac{\mathrm{d}y}{\sqrt{a^3 - 3ay^2 - 2y^3}} = -\frac{2}{\sqrt{3a}}\,\mathrm{arctanh}\left(\sqrt{\frac{a-2y}{3a}}\right).$$

$*$10. *An example of a regular expansion.* The flow down a slightly corrugated channel is described by a function $u(x, y; \epsilon)$ which is periodic in x and which satisfies

$$\nabla^2 u = -1 \quad \text{in} \quad |y| \leqslant h(x;\epsilon) \equiv 1 + \epsilon \cos kx\ ,$$
$$\text{subject to} \quad u = 0 \quad \text{on} \quad y = \pm h(x;\epsilon)\ .$$

Obtain the first three terms for u and hence evaluate correct to $\mathrm{ord}(\epsilon^2)$ the average flux per unit width

$$\frac{k}{2\pi}\int_{x=0}^{2\pi/k}\int_{y=-h(x;\epsilon)}^{+h(x;\epsilon)} u(x, y; \epsilon)\, dx dy\ .$$