3C7d **PSM: Examples Sheet 4** Michaelmas 2008

A star $*$ denotes a question, or part of a question, that should not be done at the expense of unstarred questions (e.g. you might like to miss them out first time through the sheet). You are welcome to use an algebraic manipulator if you think it would help. Corrections, suggestions and comments should be emailed to S.J.Cowley@damtp.cam.ac.uk.

1. By means of Watson's lemma, show that the asymptotic expansion of the integral

$$F(z) = \int_1^\infty \exp(-z^3 t^3)\,\mathrm{d}t$$

for real $z \gg 1$, is

$$F(z) \sim \frac{\exp(-z^3)}{3z^3} \sum_{r=0}^{\infty} \frac{\Gamma(r+\frac{2}{3})}{\Gamma(\frac{2}{3})} \frac{1}{(-z^3)^r}\,.$$

Hence find an asymptotic expansion, including exponentially small terms, for

$$G(z) = \int_0^1 \exp(-z^3 t^3)\,\mathrm{d}t\,.$$

Now suppose that z is complex. By considering contours of steepest descent, find the asymptotic expansions of $G(z)$ for $0 \leqslant \arg(z) \leqslant 2\pi$ away from Stokes lines, which should be identified. Selected contours of constant $\Re(-z^3t^3)/|z^3|$ for complex z are plotted in figure 1 (see overleaf), together with contours of $\Im(-z^3t^3)/|z^3|$ passing through $t = 0$ and $t = 1$.

Obtain an expression for the smoothing out of the jump in the sub-dominant term at one of Stokes lines; confirm that your result is consistent with the asymptotic expansions away from that Stokes lines obtained earlier.

You may quote the following results.

(a) The Borel sum of

$$\sum_{p=0}^{\infty} \frac{\Gamma(\gamma+p)e^{\lambda}}{\lambda^{p+\gamma}}\,,$$

for real $\gamma \geqslant 0$ and $\Re(\lambda) > 0$, is

$$I(\lambda,\gamma) = \int_0^\infty \frac{t^{\gamma-1}\exp(\lambda(1-t))}{1-t}\,\mathrm{d}t\,,$$

where the contour of integration is assumed to pass just above the pole at $t = 1$.

(b) If $\gamma \gg 1$ and

$$\lambda \sim \gamma + \mathrm{i}\mu\gamma^{\frac{1}{2}} + \nu + \dots\,,$$

where $\mu = O(1)$ and $\nu = O(1)$ then

$$I(\lambda,\gamma) \sim i\pi\left(1 + \mathrm{erf}\left(\frac{\mu}{\sqrt{2}}\right)\right)$$

(c)

$$\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right) = \tfrac{2}{\sqrt{3}}\pi\,.$$

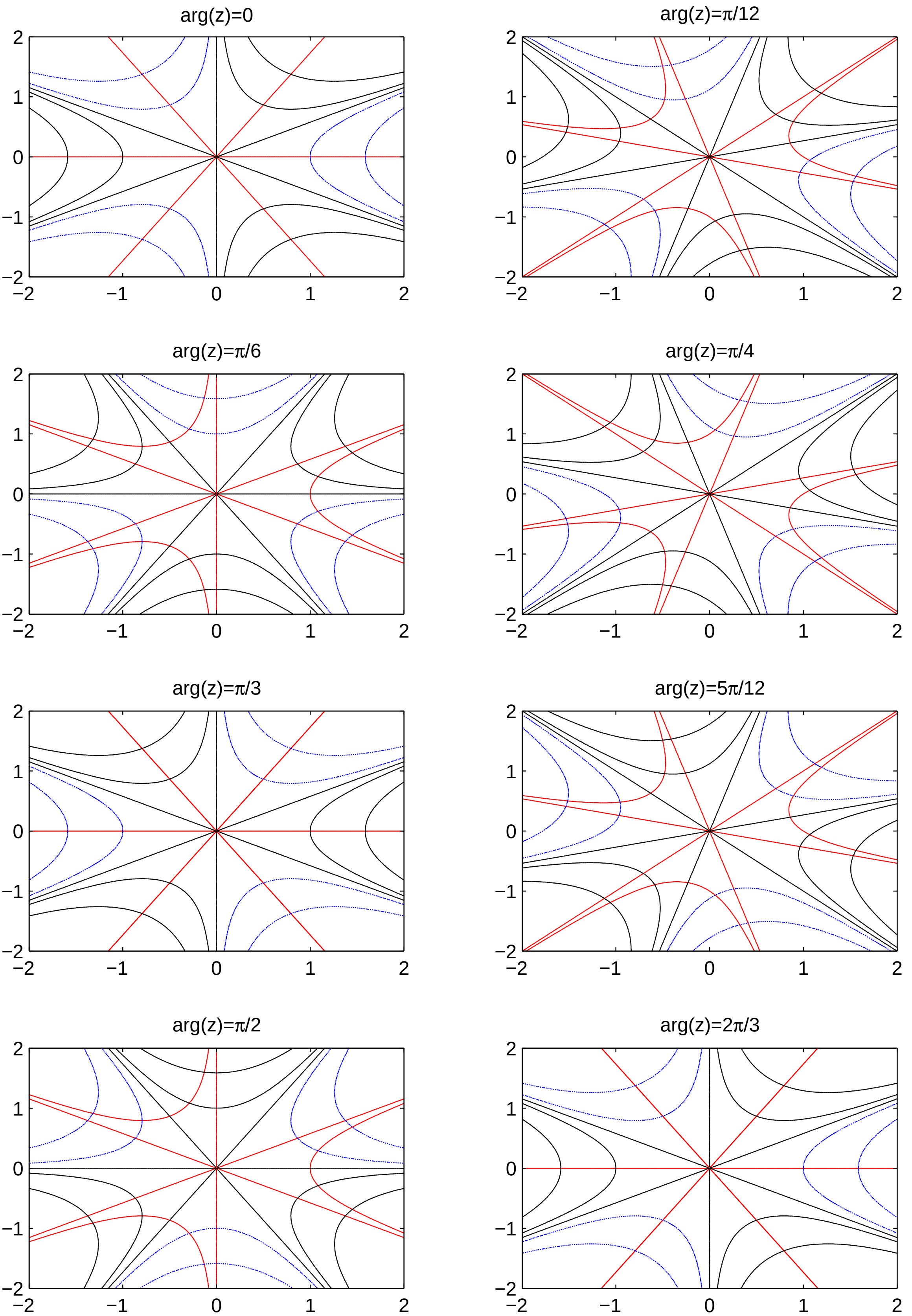

Figure 1. Contours in the complex t-plane of constant $\Re(-z^3t^3)/|z^3|$ for given complex z (black or blue depending on whether the real part is positive or strictly negative), plus contours of $\Im(-z^3t^3)/|z^3|$ passing through $t = 0$ and $t = 1$ (red).

2. Let the adjoint of the Orr-Sommerfeld operator, $\bar{\mathcal{L}}^{\dagger}_{\rm os}$, be defined so that

$$\int_{-1}^{1} (\bar{\mathcal{L}}_{\rm os}\phi)^* \psi \, dy = \int_{-1}^{1} \phi^* \bar{\mathcal{L}}^{\dagger}_{\rm os} \psi \, dy,$$

for all functions $\phi(y)$ and $\psi(y)$ such that

$$\phi = \phi' = \psi = \psi' = 0 \quad \text{on} \quad y = \pm 1\,,$$

where * denote a complex conjugate. Show that the frequency eigenvalues of the adjoint Orr-Sommerfeld operator are the complex conjugates of those of the direct operator.

For the inner product $(\phi, \psi)_E$ defined by

$$(\phi, \psi)_E = \int_{-1}^{1} \phi^* \left(\frac{d^2}{dy^2} - \alpha^2 - \beta^2 \right) \psi \, dy,$$

show that eigenfunctions of the Orr-Sommerfeld operator and its adjoint are mutually orthogonal.

3. *Squire modes are stable.* The [homogeneous] Squire equation is

$$\bar{\mathcal{L}}_{\rm sq}\widetilde{\eta} \equiv (-\mathrm{i}\omega + \mathrm{i}\alpha U)\widetilde{\eta} - \frac{1}{R}\left(\frac{d^2}{dy^2} - \alpha^2 - \beta^2 \right) \widetilde{\eta} = 0\,,$$

where $\widetilde{\eta}$ is subject to $\widetilde{\eta}(\pm 1) = 0$. Show that the eigensolutions of this equation are temporally stable.

4. *Bénard convection.* There are no simple exact solutions of the Orr-Sommerfeld equation. However, in the case of the thermal instability of a viscous fluid it is possible to find exact linear stability solutions. We study a layer of fluid heated from below, e.g. oil in a frying pan.*

First some background.† Suppose that the region $0 \leqslant \bar{z} \leqslant d$ is occupied by a viscous fluid, and that the lower and upper boundaries are maintained at temperatures T_0 and T_1 respectively. Under the Boussinesq approximation, the governing equations for the fluid are then

$$\begin{aligned} \frac{\partial \bar{\mathbf{u}}}{\partial \bar{t}} + \bar{\mathbf{u}} \cdot \boldsymbol{\nabla} \bar{\mathbf{u}} &= -\frac{1}{\rho} \boldsymbol{\nabla} (\bar{p} + \rho g \bar{z}) + \alpha g (\bar{T} - T_0) \widehat{\mathbf{z}} + \nu \boldsymbol{\nabla}^2 \bar{\mathbf{u}}\,, \\ \boldsymbol{\nabla} \cdot \bar{\mathbf{u}} &= 0\,, \\ \frac{\partial \bar{T}}{\partial \bar{t}} + \bar{\mathbf{u}} \cdot \boldsymbol{\nabla} \bar{T} &= \kappa \boldsymbol{\nabla}^2 \bar{T}\,, \end{aligned}$$

where $\bar{\mathbf{u}}$, $\bar{T}$, $\bar{p}$, ρ, ν, κ and α are the velocity, temperature, pressure, density, viscosity, thermal diffusivity and coefficient of expansion of the fluid respectively, while g is gravity and $\widehat{\mathbf{z}}$ is the unit vector in the $\bar{z}$-direction. An exact solution of the governing equations is

$$\bar{\mathbf{u}} = 0\,, \quad \bar{T} = T_0 - \beta \bar{z}\,, \quad \bar{p} = p_0 - \rho g \bar{z} \left(1 + \tfrac{1}{2} \alpha \beta \bar{z} \right)\,,$$

where $\beta = (T_0 - T_1)/d$.

To test the stability of this steady solution write

$$\bar{\mathbf{u}} = \frac{\kappa}{d} \mathbf{u}\,, \quad \bar{T} = T_0 - \beta \bar{z} + \beta d T\,, \bar{p} = p_0 - \rho g \bar{z} \left(1 + \tfrac{1}{2} \alpha \beta \bar{z} \right) + \frac{\rho \kappa^2}{d^2} p\,,$$

where $\mathbf{u}$, T and p are assumed to be infinitesimally small. If we also non-dimensionalise space and time by

$$\bar{\mathbf{x}} = d\mathbf{x}\,, \quad \bar{t} = \frac{d^2}{\kappa} t\,,$$

then on substituting into the governing equations and linearising it follows that

$$\frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\nabla} p + \sigma R T \widehat{\mathbf{z}} + \sigma \boldsymbol{\nabla}^2 \mathbf{u}\,, \tag{1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0\,, \tag{2}$$

$$\frac{\partial T}{\partial t} - w = \boldsymbol{\nabla}^2 T\,, \tag{3}$$

* Strictly this application is a bit of a cheat, since surface tension effects that we neglect are in fact important.

† There is no need to check this unless you want to. The question starts after equation (5).

where $\mathbf{u} = (u, v, w)$, $\sigma = \nu/\kappa$ is the *Prandtl number*, and $R = g\alpha\beta d^4/\kappa\nu$ is the *Rayleigh number*. Two of the boundary condition are straightforward, i.e.

$$w = 0 \quad \text{and} \quad T = 0 \quad \text{on } z = 0 \text{ and } z = 1. \tag{4}$$

In order to simplify the analysis, also assume stress-free conditions at the boundaries, i.e.

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad \text{on } z = 0 \text{ and } z = 1. \tag{5}$$

Show from (1) and (2) that

$$\frac{\partial}{\partial t}\nabla^2 w = \sigma R\,\nabla_{\mathrm{h}}^2 T + \sigma \boldsymbol{\nabla}^4 w\,, \tag{6}$$

where $\nabla_{\mathrm{h}}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian. Also deduce that

$$D^2 w = 0 \quad \text{on } z = 0 \text{ and } z = 1, \text{ where } D = d/dz. \tag{7}$$

On the basis that (3), (4), (6) and (7) have coefficients that are independent of $\mathbf{x}$ and t seek solutions of the form

$$w = W(z)f(x,y)e^{st}\,, \quad T = \theta(z)f(x,y)e^{st}\,,$$

where

$$\nabla_{\mathrm{h}}^2 f + k^2 f = 0\,.$$

Deduce that

$$\begin{aligned} (D^2 - k^2)\left(D^2 - k^2 - \frac{s}{\sigma}\right)W &= k^2 R\,\theta\,, \\ (D^2 - k^2 - s)\theta &= -W\,, \end{aligned}$$

and that $W = D^2W = \theta = 0$ on $z = 0$ and $z = 1$. On the assumption that the eigensolutions for W are

$$W_n = \sin n\pi z\,,$$

find the dispersion relation. Deduce that if $R < 0$ then $\Re(s) < 0$, and that if $R > 0$ then $\Im(s) = 0$. Find the critical Rayleigh number R_c such that all modes are stable for $R < R_c$ and at least one mode is unstable for $R > R_c$.

5. Consider the inviscid flow, $(U(y), 0, 0)$, of a stratified fluid with density $\rho(y)$. Assume that there are rigid walls at $y = -1$ and $y = 1$. In the so-called *Boussinesq limit*, it may be shown that the equation governing linear two-dimensional perturbations to this flow profile and density profile is

$$(U - c)\left(D^2 - \alpha^2\right)\phi - U''\phi + \frac{J(y)\phi}{U - c} = 0\,,$$

where $D = \frac{d}{dy}$, $U'' = \frac{d^2U}{dy^2}$,

$$\mathbf{u} = (U(y), 0, 0) + \left(\frac{d\phi}{dy}, -i\alpha\phi, 0\right) exp(i\alpha(x - ct)) + \dots\,,$$

and

$$J(y) = -\frac{1}{\rho}\frac{d\rho}{dy}\,.$$

If $H = (U - c)^{-\frac{1}{2}}\phi$, show that

$$D((U - c)DH) - \left(\alpha^2(U - c) + \tfrac{1}{2}U'' + \frac{\tfrac{1}{4}U'^2 - J}{U - c}\right)H = 0\,.$$

Hence deduce that if the flow is unstable, then somewhere in the flow

$$J < \tfrac{1}{4}U'^2\,.$$

6. *The inviscid stability of the velocity profile $u_0 = \sin y$.*

   Suppose inviscid fluid flows between walls at $y = y_1$ and $y = y_2$ with velocity profile $u_0 = \sin y$. Is Fjortoft's criterion satisfied if $y_1 < 0 < y_2$? By solving Rayleigh's equation show that no neutral modes can be found if

$$(y_2 - y_1) < \pi \ .$$

   If $(y_2 - y_1) > \pi$, comment on the existence of modes for

$$\alpha < \alpha_s \ \text{ and } \ \alpha > \alpha_s \ ,$$

   where α_s is the wavenumber of a neutral mode.

7. Solve Rayleigh's equation for the linear mixing layer profile

$$U(y) = \begin{cases} 1 & y \geqslant 1 \\ y & -1 \leqslant y \leqslant 1 \\ -1 & y \leqslant -1 \end{cases} .$$

   For what ranges of real k is the flow temporally unstable? Can you say anything about the possibility of absolute instability?

   Hint: use the fact that the normal velocity and the pressure are continuous, with the pressure being proportional to $(\omega - kU)\widetilde{u} + i\widetilde{v}(dU/dy)$ and with $(\widetilde{u}, \widetilde{v})$ being the perturbation velocity.

8. Consider inviscid fluid flow between rigid walls at $y = -1$ and $y = 1$. Initially the velocity profile is given by $(U(y), 0, 0)$. Suppose now that the flow is perturbed so that

$$\mathbf{u} = (U(y), 0, 0) + (u, v, w) \, .$$

   Derive linearised governing equations for the velocity v and the vorticity $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$.

   Suppose that the perturbations have a single Fourier component so that

$$(v, \eta) = (\widetilde{v}(y,t), \widetilde{\eta}(y,t)) exp(i(\alpha x + \beta z)) \, ,$$

   and let $\widetilde{v}_0(y) = \widetilde{v}(y, 0)$ and $\widetilde{\eta}_0(y) = \widetilde{\eta}(y, 0)$. If $U = \lambda y$ and $\alpha \neq 0$, then by means of a Laplace transform, or otherwise, find an integral expression for $\widetilde{v}$. Also find an an integral expression for $\widetilde{\eta}$ in terms of $\widetilde{v}$.

   Consider separately the case when $\alpha = 0$ and $\beta \neq 0$. Solve for $\widetilde{\eta}$, and comment on your result.

9. Consider

$$\frac{d\mathbf{q}}{dt} = \mathsf{A}\mathbf{q} \, .$$

   Calculate upper and lower bounds on the maximum possible growth at time t, extremised over all nonzero initial conditions, for each of the following matrices:

$$\mathsf{A} = \begin{pmatrix} -\frac{1}{R} & 0 \\ 1 & -\frac{2}{R} \end{pmatrix} , \qquad \mathsf{A} = \begin{pmatrix} -\frac{i}{R} & 0 \\ 1 & -\frac{i}{R} \end{pmatrix} , \qquad \mathsf{A} = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{2}{R} \end{pmatrix} .$$

   Determine the resolvent matrix, and attempt to describe the pseudo-spectra, in each case.

10. Consider the linear Kelvin-Gordon equation

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 A = \mu A + \frac{\partial^2 A}{\partial x^2} \, .$$

   Find the location of any points at which the group velocity is zero. For $U = 2$ and $\mu = 1$ is the flow absolutely unstable? Discuss other values of U and μ.

11. In two dimensions the linearised Ginzburg-Landau equation with constant coefficients is

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} + V\frac{\partial \eta}{\partial y} - \mu\eta - \gamma_{11}\frac{\partial^2 \eta}{\partial x^2} - \gamma_{22}\frac{\partial^2 \eta}{\partial y^2} - 2\gamma_{12}\frac{\partial^2 \eta}{\partial x \partial y} = 0 \ ,$$

where U, V, μ and γ_{ij} $(i, j = 1, 2)$ are real constants. By considering solutions of the form $\eta = \exp(-\mathrm{i}\omega t + \mathrm{i}\alpha x + \mathrm{i}\beta y)$ and computing the group velocity, find a necessary condition for the occurrence of absolute instability in the form $\mu > f(U, V)$, where $f(U, V)$ is a quadratic function of U, V to be determined.

If (U, V) represents a mean-flow velocity whose magnitude is fixed but whose orientation can be changed, i.e. $(U, V) = W(\cos\theta, \sin\theta)$ for some fixed W, find the minimum value of μ required to guarantee the absence of absolute instability for *any* mean-flow direction.

$*$12. Consider the linearised Ginzburg-Landau equation with slow spatial variation of the growth parameter μ, i.e.

$$\frac{\partial A}{\partial t} + U\frac{\partial A}{\partial x} = \mu(X)A + \gamma\frac{\partial^2 A}{\partial x^2} \ ,$$

where $X = \epsilon x$ and $\epsilon \ll 1$. Seek a solution of the general form

$$A = (Q_0(X) + \epsilon Q_1(X) + \dots)\exp\left(-i\omega t + \frac{i}{\epsilon}\int_0^X k(\xi)\, d\xi\right) \ ,$$

and find an expression for Q_0.

$*$13. Consider the linearised Ginzburg-Landau equation with $\mu(X) = \mu_0 - \nu X^2$ and other coefficients constant. We look for a *global mode* by setting $A = a(x)\exp(-i\omega_G t)$ and searching for solutions which are bounded as $x \to \pm\infty$. Show, by writing $a(x) = \exp(Ux/2\gamma)b(\xi)$, where $\xi = \sqrt{\epsilon}cx$ for suitable c, that the global mode satisfies

$$\frac{d^2 b}{d\xi^2} + (\lambda - \xi^2)b = 0 \ , \qquad (8)$$

for suitable λ, to be found. By seeking a solution of (8) in the form $\exp(-\xi^2/2)H_n(\xi)$, where H_n is a power series, show that the solutions of (8) grow exponentially as $X \to \pm\infty$, unless $\lambda = (2n + 1)$ for any positive integer n. Hence, determine the allowed global mode frequencies and comment on the global stability of the system.