Classical Wave Scattering — Mathematical Tripos, Part III

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Example Sheet 1

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1. Consider a quiescent fluid (no mean flow) with an arbitrary temperature distribution, and take sound as a small-amplitude perturbation of an ambient state $(\rho_0, p_0, \mathbf{v}_0)$, Starting from the linearised equations for the conservation of mass and momentum and the constitutive equation for isentropic flow

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt} \ , \quad \text{where } \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \ , \tag{1}$$

derive the wave equation for the pressure in the presence of a temperature gradient

$$\frac{\partial^2 p'}{\partial t^2} - \nabla \cdot (c_0^2 \nabla p') = 0. \tag{2}$$

Consider temperature gradients over a small height, so that the variation in p_0 can be neglected (since $\nabla p_0 / p_0 \ll \nabla T_0 / T_0$)

(a) What is the main difference between this wave equation, and the wave equation for the pressure in the case of a quiescent fluid with no temperature gradient?

(b) For a quiescent fluid with no temperature gradient, do the acoustic pressure, the density, and the velocity potential all satisfy the same wave equation?

(c) In the same case as (b), is it sufficient to work with any one acoustic variable, and solve the wave equation for it, in order to derive solutions for the other acoustic variable?

(d) In the case of a quiescent fluid with a temperature gradient, are the answers to questions equivalent to (b) and (c) the same?

2. Consider a time-harmonic, monochromatic plane wave incident at an angle θ onto an interface which has surface impedance Z. Writing the total field above the interface as the sum of an incident and a reflected field, express the reflection coefficient in terms of the impedance.

(a) Write the mean (time-averaged) energy flux across the surface in terms of the impedance, and find a condition on the real part of the impedance which holds if the surface absorbs energy.

3. Consider a time-harmonic, monochromatic plane wave in a 2-dimensional space (x, z), incident at an angle θ_1 upon an impedance surface defined by the x-axis: $z = 0$, which divides the space into an upper medium with density ρ_1, wavespeed c_1 and corresponding wavenumber $k_1 = \omega/c_1$, and a lower medium with density ρ_2, wavespeed c_2 and corresponding wavenumber $k_2 = \omega/c_2$. Suppose that a perfectly reflecting surface is placed at $z = -d$, and impose Dirichlet boundary conditions at the surface $z = -d$.

   Derive an expression for the total acoustic field at an arbitrary point $(x, -d < z < 0)$ in medium 2.

4. Consider an infinite waveguide in a 3-dimensional space, defined by 2 infinite parallel plates at $z = 0$ and $z = a$, and a point source inside the wave guide, at a point (x_0, y_0, z_0), with $0 < z_0 < a$. Given that the plates are perfectly reflecting, with Dirichlet boundary conditions,

   (a) find the Green's function inside the wave guide [use the method of images].

   (b) give an expression for the total field at an arbitrary point inside the wave guide.

5. Consider an infinite string in a 2-dimensional space. The equilibrium position of the string coincides with the x-axis: $z = 0$, and the string has density ρ_1 for $x < 0$ and ρ_2 for $x > 0$. An incident wave from $x = -\infty$ is

$$\psi_{inc} = e^{(ik_1 x - i\omega t)} \ .$$

   Then

$$\begin{aligned} \psi &= e^{(ik_1 x)} + \psi_-(x) \qquad \text{for } x < 0 \\ \psi &= \psi_+(x) \qquad \text{for } x < 0 \end{aligned}$$

   and the governing equations are:

$$\frac{d^2\psi_-}{dx^2} + k_1\psi_- = 0 \qquad \text{for} \quad x < 0 \tag{3}$$

$$\frac{d^2\psi_+}{dx^2} + k_2\psi_- = 0 \qquad \text{for} \quad x < 0 \ , \tag{4}$$

where $k_i^2 = \omega^2 \rho_i / T$, $T =$ tension. The boundary conditions are:

$$\psi_- \quad \sim Re^{(-ik_1 x)} \qquad \text{as } x \to -\infty \tag{5}$$
$$\psi_+ \quad \sim Te^{(ik_2 x)} \qquad \text{as } x \to \infty \tag{6}$$
$$(\psi_+ \quad + \psi_-) = 1 \ , \quad (\psi'_+ + \psi'_-) = ik_1 \quad \text{at } x = 0 \ . \tag{7}$$

(a) Write a Wiener-Hopf equation for the (unknown) transforms of ψ_+ and ψ_- [assign a small imaginary part to k_1 and k_2].

(b) Use the Wiener-hopf technique to find the solutions ψ_+ and ψ_- .

6. Find the energy flux associated with the forward-going solution obtained with the WKB approximation in an inhomogeneous medium. What implications does it have for energy conservation?