Example Sheet 2

Corrections and comments to O.Rath-Spivack@damtp.cam.ac.uk

1. (Statistics revision) Let ϕ_i be a set of random variables, uniformly distributed in $[0, 2\pi)$, with ϕ_i, ϕ_j independent for $i \neq j$. Denote averages by angled brackets $< ... >$.

   (a) Show (using the probability distribution function or otherwise)that for any $x,\ y$

$$\begin{aligned} < sin(x+\phi_n) > &= 0 \\ < sin(x+\phi_n) >< cos(y+\phi_n) > &= 0 \\ < sin^2(x+\phi_n) > &= \frac{1}{2} \\ < sin(x+\phi_n) >< sin(y+\phi_n) > &= \frac{1}{2}\cos\xi \end{aligned}$$

   where $\xi = y - x$.

   (b) Describe how to generate a continuous random surface $f(x)$ with a given autocorrelation function, as a superposition of N sine waves with random phases $\phi_i, i = 1, ..., N$.

   (c) If ϕ is a random variable with normal distribution, mean zero and variance $< \phi^2 >= \phi_0$, find the mean value $< \exp(i\phi) >$.

2. Let $f(x)$ be a continuous stochastic process which is statistically stationary and with autocorrelation function

$$\rho(\xi) =< f(x)f(x+\xi) > \quad .$$

   Show that

$$\left\langle f(y)\frac{df(x)}{dx}\right\rangle = \left.\frac{d\rho}{d\xi}\right|_{\xi=y-x} .$$

   and similarly find an expression for the autocorrelation of the slope

$$\left\langle \frac{df(x_1)}{dx}\frac{df(x_2)}{dx}\right\rangle$$

   For what value of ξ does ρ reach a global maximum? What happens as $\xi \to \infty$?

3. Consider a field $\phi e^{i\omega t}$ in a weakly scattering extended random medium (x, z), where x is horizontal and z is vertical. Let the refractive index be $n(x, z) = 1 + \mu W(x, z)$ where W is statistically stationary, has mean zero, and variance one, with autocorrelation function

$$\rho(\xi, \eta) = < W(x, z)W(x + \xi, z + \eta) >$$

and scale sizes H and L in the horizontal and vertical direction respectively. Suppose that the slowly varying part of the field E obeys the parabolic equation. Over a small distance d the effect of the medium on the field is a phase change $E(x + d, z) = E(x, z)e^{i\phi}$.

(a) Show that ϕ can be written

$$\phi(z) = k_0\mu \int_x^{x+d} W(x', z)dx'$$

and find its variance $< \phi^2 >$.

(b) Describe the effect on the scattering of 'stretching' each of the scale sizes H and L.

4. Consider a region V_1 contained between two smooth closed surfaces S_0 and S_1. Here the surface S_0 encloses a source-free volume V_0, and the volume V_1 contains the closed surface S_0 and a source $\mathbf{J}(\mathbf{r})$.

Starting from the Helmholtz (vector) wave equation for the electromagnetic field in V_1 and in the source-free region:

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E_1}(\mathbf{r})) - k_1^2\mathbf{E_1}(\mathbf{r})) &= i\omega\mu\mathbf{J}(\mathbf{r}) \\ \nabla \times (\nabla \times \mathbf{E_2}(\mathbf{r})) - k_2^2\mathbf{E}(\mathbf{r})) &= 0 \end{aligned}$$

and using the dyadic Green's function, derive a vector integral equation for the electric field in V_1, equivalent to the Kirchoff-Helmholtz equation derived for the acoustic field. (*Use Gauss theorem, and let* $S_1 \to \infty$.)

5. Consider a 2D infinite, flat interface, at $z = z_0$ separating the 3D space into two *anisotropic* half-spaces. Denote by $\bar{\epsilon}_1$ and $\bar{\mu}_1$ the permittivity and permeability in medium 1, and similarly by $\bar{\epsilon}_2$ and $\bar{\mu}_2$ the equivalent quantities in medium 2.

A known, plane electromagnetic wave is incident upon the interface, travelling from medium 1 towards medium 2.

(a) Write the state equation and use it to write the general form of the state vector:

$$\mathbf{V}(z) = \bar{\mathbf{a}} \cdot e^{i\bar{\beta}} \cdot \mathbf{A}$$

Comment on the quantities $\bar{\mathbf{a}}$, $\bar{\beta}$, and $\mathbf{A}$ and their dimensions.

(b) Using appropriate boundary conditions, write equations that relate upgoing and downgoing waves in the two regions, and find expressions for the reflection and transmission matrix, given that the incident wave is known.

6. Consider a time-harmonic, monochromatic plane wave in a 2-dimensional space (x, z), incident at an angle θ_1 upon an impedance surface defined by the x-axis: $z = 0$, which divides the space into an upper medium with density ρ_1, wavespeed c_1 and corresponding wavenumber $k_1 = \omega/c_1$, and a lower medium with density ρ_2, wavespeed c_2 and corresponding wavenumber $k_2 = \omega/c_2$. Suppose that a perfectly reflecting surface is placed at $z = -d$, and impose Dirichlet boundary conditions at the surface $z = -d$.

   Derive an expression for the total acoustic field at an arbitrary point $(x, -d < z < 0)$ in medium 2.

7.