Classical Wave Scattering
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Mathematical Tripos, Part III
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Example Sheet 3

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1. Consider a 3-dimensional, random medium with refractive index

$$n = 1 + \mu W(x, y, z) \ ,$$

where μW is a normally distributed random correction to the refractive index, such that $< W > \;\; = 0$ and $< W^2 > \;\; = 1$, and $\mu \ll 1$ is a small constant. Consider a time-harmonic field which is assumed to be propagating at a small angle to the horizontal x.

(a) derive the equation giving the evolution along x of the first moment $< E >$, where E is the reduced wave.

(b) Assume the medium is statistically stationary in y and z. Find the average field at an arbitrary point (x, y, z) due to a plane wave propagating in the positive x direction and defined by $E(x, y, z) = E_0$ at $x = 0$.

(c) What happens if the field is a superposition of plane waves, for example a Gaussian beam?

2. (a) Formulate the Kirchhoff (tangent plane) approximation for scattering from a rough surface in the case of electromagnetic waves.

Hint: start from the integral Kirchoff-Helmholtz for vector fields, expressed in terms of a scalar Green's function, then consider the far-field approximation and use appropriate expressions for the Green's function $G(\mathbf{r}, \mathbf{r}')$ *and for* $\nabla G(\mathbf{r}, \mathbf{r}')$*, then approximate the surface fields using Snell's law applied to the tangent plane and the local plane of incidence. You will need the vector identity* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

3. The vertical plane (y, z) divides the space into medium 1 for $x < 0$ and medium 2 for $x \geq 0$. We take medium 1 to be free space, so $\mathbf{J} = 0$, and we denote its permittivity by ϵ_0 and its permeability by μ_0. Medium 2 is a conductive medium, with $\mathbf{J} = \sigma\mathbf{E}$, permeability $\mu = \mu_0$, and we write its permittivity as

$$\tilde{\epsilon} = \epsilon - i\frac{\sigma}{\omega} \qquad (3.1)$$

where ϵ is the usual, real permittivity of the medium, and ω is the field frequency.

(a) Use Maxwell's equations to show how a complex permittivity $\tilde{\epsilon}$ is naturally defined.

(b) An electromagnetic plane wave propagating in the positive x direction from free space is normally incident on the interface. Assuming that the wave is vertically polarized, with electric field given by

$$\mathbf{E_{inc}} = \hat{\mathbf{z}} E_0 e^{-ikx} \ ,$$

where $k = \omega\sqrt{\mu\epsilon}$,

(i) find the total electric and magnetic field for $x < 0$ and for $x \geq 0$;

(ii) find the power associated with the wave in free space, and the power transmitted into the conducting medium;

How does the transmitted power depend on the frequency of the incident wave?

4. Consider a plane electromagnetic wave travelling in the positive x direction in free space, and normally incident on a screen at $x = 0$, which contains an aperture A. The incident wave is linearly polarized, and can be written as

$$\mathbf{E_{inc}} = \psi(x, y, z)\mathbf{p}(x, y, z) \ ,$$

where $\mathbf{p}(x, y, z) = (0, 1, 0)$ is a vector and $\psi(x, y, z)$. The field at $x = 0$ is defined to be $\mathbf{E_{inc}}$ on the aperture, and zero everywhere else.

(a) Find vectors $\mathbf{P}(\nu, \omega)$ and constants $\alpha(\nu, \omega)$, where

$$(\mathbf{n} \times \mathbf{p}) = \alpha(\nu, \omega)(\mathbf{n} \times \mathbf{P}(\nu, \omega)$$

such that the field at $x > 0$ can be written as

$$\mathbf{E}(x, y, z) = \int \alpha(\nu, \omega)\hat{\psi}(\nu, \omega)e^{iq(\nu,\omega)x}e^{i(\nu y + \omega z)}\mathbf{P}(\nu, \omega)d\nu d\omega$$

Here $\hat{\psi}(\nu, \omega)$ is the 2-dimensional discrete Fourier transform of the scalar part of the field at $x = 0$, defined by

$$\psi(0, y, z) = \int \hat{\psi}(\nu, \omega) e^{i(\nu y + \omega z)} d\nu d\omega$$

so that

$$\mathbf{E}(0, y, z) = \left(\int \psi \hat{\psi}(\nu, \omega) e^{i(\nu y + \omega z)} d\nu d\omega \right) \mathbf{p} \ .$$

Hint: use the fact that the field beyond the aperture can be written as

$$\mathbf{E}(x) = \nabla \times \int_A (\mathbf{n} \times \mathbf{E}_{inc}(x, y, z)) G dy dz \quad ,$$

where G is the scalar free space Green's function.

(b) How does this problem, and the Fourier decomposition, differs for electromagnetic waves as opposed to acoustic waves?

5. Consider the inverse problem of finding the permittivity $\epsilon(\mathbf{r})$ of an inhomogeneity, given an incident field $\mathbf{E}_{inc}(\mathbf{r})$ generated by a point source ar $\mathbf{r}_0$, and a measured scattered field $\mathbf{E}_s(\mathbf{r})$.

   (a) write an expression for $\mathbf{E}_s(\mathbf{r})$ to first order in the distorted wave Born approximation.

   (b) discuss some reasons why this inverse problem is ill-posed.