Symmetric Dynamical Systems **J.H.P. Dawes**
Part III 2007–08

Example Sheet 1: Dynamics in $\mathbb{R}^2$; bifurcations in 1D ODEs and 1D maps.

Starred questions are not necessarily harder, just less central to subsequent course material. Send comments and queries to `J.H.P.Dawes@damtp.cam.ac.uk`.

1. Consider the linear system $\dot{x} = Ax$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $T = a + d$ and $D = ad - bc$. Indicate the regions in the (T, D)–plane where the origin is a stable node, stable focus, unstable focus, unstable node and saddle point. In which regions is the origin stable?

2. Using the results of question 1, sketch the phase portrait of the following linear systems in $\mathbb{R}^2$:

   (a) $\dot{x} = -2x - 3y$, $\quad \dot{y} = 8x + 8y$;
   (b) $\dot{x} = x$, $\quad \dot{y} = -3x - y$;
   (c) $\dot{x} = 7x - 2y$, $\quad \dot{y} = 5x + 5y$;
   (d) $\dot{x} = 7x + 5y$, $\quad \dot{y} = -10x - 7y$;
   (e) $\dot{x} = 2x + y$, $\quad \dot{y} = 2y$;
   (f) $\dot{x} = 4x + 2y$, $\quad \dot{y} = 2x + y$;
   (g) $\dot{x} = 8x + 3y$, $\quad \dot{y} = -3x + 2y$;
   (h) $\dot{x} = -2x$, $\quad \dot{y} = -2y$.

   For which systems is the origin a hyperbolic equilibrium and for which is it stable?

3. Find all equilibria of the system

$$\begin{aligned} \dot{x} &= 2x - x^3 - 3xy^2, \\ \dot{y} &= y - y^3 - x^2 y. \end{aligned}$$

   Find their stability types (stable node, saddle point etc) and sketch the phase portrait, noting what happens on the axes and any symmetries present.

4. For the nonlinear oscillator $\ddot{x} + ax + bx^2 = 0$ show that $V = \frac{1}{2}p^2 + \frac{1}{2}ax^2 + \frac{1}{3}bx^3$ (where $p = \dot{x}$) is conserved along trajectories, i.e. $\frac{dV}{dt} = 0$. Sketch the level sets $V =$ constant in the (x, p)–plane assuming a, $b > 0$ and describe the different types of orbit in the system. By computing $\frac{dV}{dt}$ show that when $k > 0$ there can be no periodic orbits for the system $\ddot{x} + k\dot{x} + ax + bx^2 = 0$. Compute the linear stability type of each equilibrium. Sketch the complete phase portrait showing the sets of initial conditions whose trajectories converge to each of the equilibria as $t \to \infty$.

5. Consider the ODEs

$$\dot{x} = -x - \frac{y}{\log\sqrt{x^2 + y^2}}, \qquad \dot{y} = -y + \frac{x}{\log\sqrt{x^2 + y^2}}.$$

   By changing to polar co-ordinates show that trajectories of the flow repeatedly circle around the origin, even though the linearised system at the origin is that of a stable node.

6. Find the equilibrium points in the following 1D systems and determine their stability. Find the value(s) of μ at which bifurcations occur and sketch a bifurcation diagram in each case.

   (a) $\dot{x} = x(\mu - x^2)(1 - \mu + \frac{1}{2}x^2)$;
   (b) $\dot{x} = \mu - 2x^2 + x^4$;
   (c) $\dot{x} = x(\mu + 2x^2 - x^4)$;
   (d) $\dot{x} = x(x^2 - \mu)(x^2 + \mu^2 - 1)$.

7. Consider the behaviour of the two-parameter family of 1D systems $\dot{x} = \mu_1 + \mu_2 x + x^2$. Sketch in the (μ_1, μ_2) plane the regions in which the system has 0, 1 or 2 equilibria. **Use your sketch** to investigate the bifurcations which occur in the following one-parameter families:

(a) $\dot{x} = \mu + x + x^2$,
(b) $\dot{x} = 2\mu + \varepsilon - (\mu + 2)x + x^2$.

In each case sketch a bifurcation diagram showing the position and stability of the equilibria as μ varies. In (b) you should consider separately the three cases $\varepsilon = 0$, ε small and positive, and ε small and negative.

8. Find the equilibrium point of the ODEs

$$\begin{aligned} \dot{x} &= a - (b+1)x + x^2 y, \\ \dot{y} &= bx - x^2 y. \end{aligned}$$

where a and b are positive parameters. Find the set of values of a and b at which there is a bifurcation and plot this bifurcation set (curve) in the (a, b)–plane.

9. For each of the following maps, $x_{n+1} = F(x_n, \mu)$, describe the bifurcation that occurs at each given value of μ.

(a) $F(x, \mu) = (1 + \mu)x - x^2$ at $\mu = 0$,

(b) $F(x, \mu) = (1 + \mu)x - x^3$ at $\mu = 0$,

(c) $F(x, \mu) = \mu - x^2$ at $\mu = -\frac{1}{4}$ and $\mu = \frac{3}{4}$,

(d) $F(x, \mu) = \mu x - x^3$ at $\mu = -1$, $\mu = 1$ and $\mu = 2$.

Now consider the map $g(x, \mu) = F(F(x, \mu), \mu) \equiv F^2(x, \mu)$. Show that the trivial fixed point of g cannot undergo a period–doubling bifucation.

* 10. Consider the ODEs

$$\begin{aligned} \dot{x} &= \kappa x + y - x^3, \\ \dot{y} &= -\lambda x - x^3. \end{aligned}$$

Find the equilibrium points. Locate the local bifurcations of codimension one and find the conditions for the existence of a bifurcation of codimension two. Divide the (κ, λ)–plane into regions of qualitatively different behaviour and sketch the phase portraits in each region. You may assume that the Hopf bifurcation from the trivial solution is supercritical while the Hopf bifurcation from the non-trivial solution is subcritical. Speculate on what additional bifurcations are needed to make a complete description of the dynamics near the codimension two point.

* 11. Solve the following differential equations for $x(t)$ and $y(t)$:

$$\begin{aligned} \dot{x} &= -2x + y^2, \\ \dot{y} &= -y. \end{aligned}$$

Eliminate t from the solution to obtain an equation $x = g(y)$ for the trajectory passing through (x_0, y_0). Hence show that there can be no C^2 conjugacy of solutions to trajectories of the linearised system, even though (by Hartman–Grobman) they are topologically (i.e. C^0) equivalent.

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