Symmetric Dynamical Systems **J.H.P. Dawes**
Part III 2007–08

Example Sheet 3

Starred questions are not necessarily harder, just less central to subsequent course material. Send comments and queries to `J.H.P.Dawes@damtp.cam.ac.uk`.

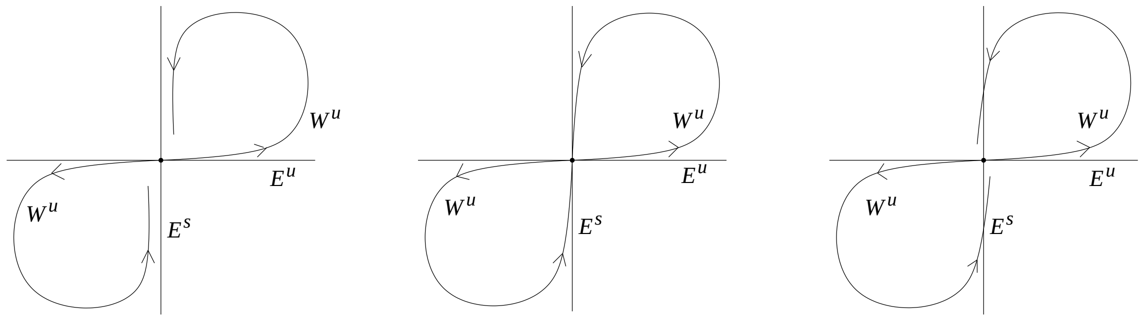
1. *The planar gluing bifurcation*. Suppose a system of ODEs

$$\begin{aligned}\dot{x} &= f(x,y,\mu),\\ \dot{y} &= g(x,y,\mu),\end{aligned}$$

has a symmetry: $f(-x,-y,\mu) = -f(x,y,\mu)$ and $g(-x,-y,\mu) = -g(x,y,\mu)$. Show that the trivial solution $x = y = 0$ is always an equilibrium. Suppose that this equilibrium is a saddle-point with one-dimensional stable and unstable manifolds tangent to the y and x axes respectively, and that near the origin the linearised behaviour is given by

$$\begin{aligned}\dot{x} &= \lambda_+ x,\\ \dot{y} &= \lambda_- y,\end{aligned}$$

where $\lambda_+ > 0 > \lambda_-$. As μ increases through zero there is a global bifurcation as the stable and unstable manifolds intersect in a figure-of-eight configuration as shown in the pictures below. This is known as the *planar gluing bifurcation*. The symmetry implies that the positive and negative branches of the unstable manifold will intersect with the positive and negative branches of the stable manifold at the same time. In this question we construct a map that models the behaviour for small μ and interpret this in terms of periodic orbits colliding with the equilibrium at the origin. We also investigate how the behaviour differs in the cases $\delta \equiv |\lambda_-|/\lambda_+ > 1$ and $\delta < 1$.



$\mu < 0$ $\mu = 0$ $\mu > 0$

Let the sides of the small box around the origin be $\Omega = \{(x,y) : |x| = h, |y| \le h\}$ and $\Sigma = \{(x,y) : |y| = h, |x| \le h\}$, inside which we use the linear flow. From the linear flow, derive the following equations for the evolution of a trajectory from a point $(x_0, y_0) = (x_0, \pm h) \to (x_1, y_1) = (\pm h, y_1)$:

$$\begin{aligned}x_1 &= h\,\mathrm{sgn}(x_0)\\ y_1 &= h\,\mathrm{sgn}(y_0)|x_0/h|^{\delta}\end{aligned}$$

where $\mathrm{sgn}(x) = x/|x|$ gives the sign of x and $\delta = -\lambda_-/\lambda_+$. Now argue that the global part of the map, close to W^u, can be approximated by the equations:

$$x_1 > 0: \quad \text{trajectory returns to} \quad \begin{aligned} x_2 &= -\mu + Ay_1\\ y_2 &= h\end{aligned}$$

$$x_1 < 0: \quad \text{trajectory returns to} \quad \begin{aligned} x_2 &= \mu + Ay_1\\ y_2 &= -h\end{aligned}$$

where A must be positive. Show that (after a rescaling) the complete return map $(x_0, y_0) \to (x_2, y_2)$ is

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\mu\,\mathrm{sgn}(x_0) + A\,\mathrm{sgn}(y_0)|x_0|^{\delta} \\ \mathrm{sgn}(x_0) \end{pmatrix} = F \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

which is symmetric under the 180° rotation $(x, y) \to (-x, -y)$. Consider the return map in the form $(x_{n+1}, y_{n+1}) = F(x_n, y_n)$ and examine the dynamics of F in the cases $\delta > 1$ and $\delta < 1$ for μ positive and negative. Look for both period-1 and period-2 orbits and hence explain the dynamics of the flow near the global bifurcation at $\mu = 0$.

2. *Taken–Bogdanov bifurcation with $\mathbb{Z}_2$ symmetry. [This is a past exam question: 2002 paper 56, Q1.]* The normal form for a particular codimension-two bifurcation is given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\lambda x + \kappa y - x^3 - x^2 y \end{aligned} \tag{1}$$

where λ and κ are real parameters.

(a) Locate the equilibrium points of (1) and sketch the curves in the (κ, λ) plane along which local bifurcations occur, classifying each bifurcation.

(b) Use the rescaling $x = \varepsilon u$, $y = \varepsilon^2 v$, $\lambda = \varepsilon^2 \alpha$, $\kappa = \varepsilon^2 \beta$, $\tau = \varepsilon t$ to deduce ODEs for u and v which, in the limit $\varepsilon \to 0$, have the conserved quantity $H(u, v) = \frac{1}{2}v^2 + \frac{\alpha}{2}u^2 + \frac{1}{4}u^4$. Sketch contours of constant H in the (u, v) plane when $\alpha < 0$. Give the value of H which corresponds to the homoclinic orbits.

(c) By integrating around one of the homoclinic orbits for small ε, find the relation between α and β, and hence between λ and κ, at the global bifurcation. Indicate this curve on your sketch of the (κ, λ) plane from part (a).

(d) Compute the saddle index $\delta = -m_-/m_+$ at the global bifurcation, where $m_+ > 0 > m_-$ are the eigenvalues of the linearisation at the relevant saddle point. Briefly describe the dynamics near the global bifurcation in this case.

Hint: refer back to question 1!

(e) Using the result of part (d) and the fact that exactly one of the Hopf bifurcations is supercritical and the other is subcritical, sketch the phase portrait of (1) in the six regions of the (κ, λ) plane which display qualitatively distinct behaviour.

3. (a) For the 2D Shilnikov map

$$\begin{pmatrix} r_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \rho + B r_n z_n^{\delta} \cos\left(\frac{\omega}{\lambda_+}\log(z_n) + \Phi_r\right) \\ -\mu + A r_n z_n^{\delta} \cos\left(\frac{\omega}{\lambda_+}\log(z_n) + \Phi_z\right) \end{pmatrix}$$

compute the determinant of the Jacobian matrix J, and show that if $\frac{1}{2} < \delta < 1$ then $\det(J) \to 0$ as $z \to 0$ and hence deduce that stable periodic orbits will be created in the saddle-node bifurcations.

Similarly, show that if $0 < \delta < \frac{1}{2}$ then all the periodic orbits will be unstable.

(b) Compare the conclusions of part (a) with the divergence $\nabla \cdot \mathbf{f}$ of the linearised vector field $\mathbf{f}$, in the two cases $\frac{1}{2} < \delta < 1$ and $0 < \delta < \frac{1}{2}$.

* 4. *The Lorenz map.* This question explores the dynamics of the Lorenz map

$$x_{n+1} = f_L(x_n) = \mathrm{sgn}(x_n)(-\mu + |x_n|^{\delta})$$

where μ is the bifurcation parameter, and δ is the ratio of the leading eigenvalues ($\frac{1}{2} < \delta < 1$). The function $\mathrm{sgn}(x)$ takes the value $+1$ if $x > 0$, 0 if $x = 0$ and -1 if $x < 0$. x and μ are not restricted to being close to zero.

(a) For $\delta = \frac{1}{2}$ show graphically that a stable period–one orbit exists in $x > 0$ for all negative values of μ. Show algebraically that it persists for $\mu < \mu_1 = \frac{1}{4}$ and describe the bifurcation that it undergoes at $\mu = \mu_1$.

(b) Again fixing $\delta = \frac{1}{2}$, locate the symmetric period–two orbit $\{\hat{x}, -\hat{x}\}$ which exists in $\mu > 0$. Show that it undergoes a bifurcation at $\mu = \frac{3}{4}$. By considering the graph of $f_L^2(x)$, or otherwise, show that this is a subcritical pitchfork bifurcation.

(c) For any $0 < \delta < 1$, show that at $\mu = \mu^* \equiv 2^{1/(\delta-1)}$, at which point $f(0^+) = f(f(0^+)) = -\mu$,

(i) trajectories cannot escape from the interval $-\mu \leq x \leq \mu$,

(ii) there are an infinite number of unstable periodic orbits.

Remark: in the Lorenz equations the point $\mu = \mu^*$ marks the transition between the "pre-turbulence" regime and the region of existence of a stable strange attractor.

(d) Repeat the calculation of part (a) to locate the bifurcation point of the period–one orbit for general $\frac{1}{2} < \delta < 1$ and show that $\mu_1 \geq \mu^*$ with equality only if $\delta = \frac{1}{2}$.

* 5. *A codimension-two global bifurcation*. Discuss the dynamics of the 1D map

$$x_{n+1} \quad = \quad f(x_n) \equiv \mu + a x_n^{1+\varepsilon}$$

when μ and ε are both small (but both may take either sign) and $x_n > 0$.

(a) Let $0 < a < 1$. Show that there is a curve of saddle-node bifurcations asymptotically close to the curve $\mu(\varepsilon) \sim \varepsilon \mathrm{e}^{-1} a^{-1/\varepsilon}$ as $\varepsilon \to 0^-$, i.e.

$$\lim_{\varepsilon \to 0^-} \frac{\mu(\varepsilon) a^{1/\varepsilon}}{\varepsilon} \quad = \quad \mathrm{e}^{-1}.$$

Sketch the form of the map $f(x)$ for each region of the (μ, ε) plane that contains qualitatively different dynamics.

(b) Repeat the analysis of part (a) in the case $a > 1$ (it is very similar).

October 29, 2007